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DYNAMICS OF AN ORIENTED POINT AND INERTIA

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It is shown that in terms of physical vacuum theory a noninertial frame can locally be realized in two ways, each of which leads to two different expressions for the forces acting on a particle. It is also shown that in the first-kind noninertial system a torsion field brings about forces that are identical to inertial forces in mechanics.

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1. Some issues of modern physics concerned with noninertial frames.

Most of the laws of physics are formulated in inertial frames of reference, where the influence of inertial forces is theoretically reduced to zero. Researchers have thus artificially introduced a class of inertial frames.

The Lorentz transformations introduced in special relativity established a relativity of inertial frames moving relative to one another at a constant speed. Special relativity has thus expanded the time-space of classical mechanics to a four- dimensional continuum structured as a flat space with a pseudo- Euclidean metrics.

When constructing his relativistic theory of gravitation, Einstein expanded the permitted transportations of coordinates. In addition to the Lorentz transformations between reference frames that are in constant motion, his theory allows for nonlinear transformations of the world coordinates, which correspond to a transition to a locally Lorentzian frame of reference (actually, to a frame of reference traveling with an acceleration.)

In transition to a locally Lorentzian reference frame, the gravitational force is compensated for by the engendered inertial force that is proportional to the translational acceleration, so that the resultant force in the equations of motion

$$\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \tag{1}$$

(locally) vanishes. Here Γ^i_{jk} are the Christoffel symbols; x^i are the contravariant coordinates of a particle, $i, j, k = 0, 1, 2, 3$.

This enabled Einstein to formulate the principle that a homogeneous gravitational field is equivalent to a constantly accelerated reference frame: "In a gravitational field all the physical processes occur in the same manner as without a gravitational field, but in an appropriately accelerated three- dimensional reference frame ("equivalence hypothesis") [1, p.400].

However, if we take a closer look at:

(a) an inertial frame of reference in an external gravitational field,

(b) a noninertial frame in the absence of an external field,

we will find a geometrical magnitude that for these frames cannot be viewed as an equivalent one, namely the Riemann tensor R^i_{jkm}.

In fact, all the components of the Riemann tensor in an accelerated frame are zero, following the tensor law of component transformation from an inertial frame in the absence of an external field, where the Riemann tensor is identically zero:

$$R^{i'}{}_{j'k'm'} = \frac{\partial x^{i'}}{\partial x^{i}}\frac{\partial x^{k}}{\partial x^{k'}}\frac{\partial x^{j}}{\partial x^{j'}}\frac{\partial x^{m}}{\partial x^{m'}}R^{i}{}_{jkm} = 0. \tag{2}$$

On the other hand, the main postulate of general relativity is the assumption that a source of a gravitational field distorts the four-dimensional time-space, so that it is described by a Riemann metric of the general form:

$$ds^2 = g_{ij}dx^i dx^j. \tag{3}$$

According to the tensor law of the transformation of the Riemann tensor components, the above refers to both inertial and accelerated frames.

It follows that in general relativity, a four-dimensional space- time without external fields is flat in inertial and accelerated frames alike. A source of a gravitational field distorts the space-time relative to any frame, both to inertial and accelerated ones. Therefore, in the general case a homogeneous gravitational field in an inertial frame of reference is not equivalent to an inertial field in a uniformly accelerated frame, but it can be represented as such under the following conditions:

(a) within a local area of space;

(b) for weak gravitational fields, and small accelerations of a noninertial frame;

(c) for nonrelativistic particle velocities.

In 1921, Einstein formulated the task of finding the geometry of space-time in a noninertial frame of another type, namely in a rotating frame. "Space-time cannot be defined in K' as in special relativity for inertial systems. But, according to the equivalence principle, K' can also be viewed as a rest frame containing a gravitational field (a centrifugal force and a Coriolis force)"[2, p.47]. (Here K' is a frame rotating relative to an inertial frame K with a constant angular velocity, the z' axis of the K' frame coinciding with the z axis of the K frame).

Einstein thus made an attempt to extend his principle of the equivalence of an inertial field concerned with a translational acceleration and a homogeneous gravitational field to become a more global principle of the equivalence of an inertial field in a rotating frame and a gravitational field. This approach, however, has not been implemented for the following reason. It is common knowledge that the force of gravitational interaction for nonrelativistic case is independent of the velocity of a test particle, which, on the other hand, was found for a component of the inertial force in a rotating frame, the Coriolis force.

Consider a formal treatment of the geometry of a rotating frame as laid down in general relativity [3, §89].

The square of an interval in an inertial frame in Cartesian coordinates is known to have the form

$$ds^2 = (c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2. \tag{4}$$

Consider a frame rotating relative to an inertial frame with a constant angular velocity Ω, so that the rotation axis coincides with the z' and z axes. The Cartesian coordinates of a point in a rotating frame, x', y', z', are related to those in an inertial frame by the obvious kinematic equations (for a nonrelativistic case):

$$\begin{aligned} &x = x' \cos \Omega t - y' \sin \Omega t\,, \\ &y = x' \sin \Omega t + y' \cos \Omega t\,, \\ &z = z'. \end{aligned} \tag{5}$$

Substituting (5) into (4) gives the interval in the rotating frame

$$ds^2 = g_{i'k'} dx^{i'} dx^{k'}, \tag{6}$$

where the metric tensor has the form:

$$g_{i'k'} = \begin{pmatrix} 1 - \frac{\Omega^2}{c^2}(x'^2 + y'^2) & \frac{\Omega}{c} y' & -\frac{\Omega}{c} x' & 0 \\ \frac{\Omega}{c} y' & -1 & 0 & 0 \\ -\frac{\Omega}{c} x' & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{7}$$

The idea behind that reasoning is the reducing of rotations, which is a change in the mutual orientation of the frames, to relationships between world coordinates of these frames. Rotation thus reduced to translational relativity of the frames.

This method is fundamentally wrong. It is well known that for the translational relativity of a four-dimensional frame to be described one only needs the four coordinates of the origin O of that frame. For the orientation of such a frame to be changed one has to introduce six angular variables - of three spatial angles and three angles determining the orientation in three pseudo-Euclidean planes [5]. It follows that corresponding to rotational relativity is a space-time with additional degrees of freedom for rotation.

A description of rotation in a space-time constructed ignoring the additional rotational degrees of freedom of the point being aligned is essentially constrained and can hardly yield correct results, especially for effects involving noncommuting rotations.

We will again address the task of describing a frame that moves with a translational acceleration of its origin O relative to the inertial frame $v_x(t) \neq const$. It is well known [5] that such acceleration causes a four-dimensional rotation of the frame, i.e., a change in the angle

$$\theta_x(t) = arcth\frac{v_x(t)}{c}, \tag{8}$$

in the pseudo-Euclidean plane XOT with the angular velocity

$$\frac{d\theta_x(t)}{dt} = \frac{1}{1-\beta^2}\frac{1}{c}\frac{dv_x(t)}{dt}, \tag{9}$$

where $\beta = v_x/c$.

It follows that the translational acceleration of an oriented point - i.e., a body whose size can be ignored, with a four- dimensional frame rigidly linked with it - can only be given a strict treatment as its rotation in pseudo-Euclidean planes in the space-time of rotational relativity. Unlike the nonlinear transformations of coordinates in general relativity theory, rotational transportations can lead to qualitatively novel results for:

(a) a combination of various four-dimensional rotations of an oriented point (e.g., a combination of translational acceleration and a three-dimensional rotation);

(b) relativistic velocities and large accelerations of particles.

2. Motion of an oriented point in the space of absolute parallelism.

Fundamental physics evolves by extending the principle of relativity. In 1988 Shipov [6] laid down the principle of universal relativity, which establishes the relativity of accelerated rotating frames of reference.

The extension of relativity principle called for an extension of the corresponding space-time. In the theory of physical vacuum constructed by Shipov on the foundation of the principle of universal relativity [5], the space-time is a ten-dimensional manifold consisting of the four-dimensional Riemannian space of world coordinates and the six-dimensional manifold of nonholonomic angular coordinates.

In fact, a four-dimensional frame of reference that is rigidly linked to an oriented point can be described by four coordinates of the frame's origin and six coordinates that describe the orientation of the origin in a four-dimensional space: three angles of spatial orientation and three pseudo-Euclidean angles that describe the orientation of the frame in three space-time planes. This space-time can feature a Riemannian curvature and a geometric torsion due to the angular coordinates being nonholonomic.

Because the full-curvature tensor of that space-time is zero, and hence a parallel translation of an arbitrary vector over a closed contour brings it to a position parallel to the initial one, this space is a geometry of absolute parallelism.

The core of the space-time of physical vacuum featuring an absolute-parallelism geometry is the field of nonholonomic tetrads consisting of the field of contravariant vectors $e^i{}_a$ and the field of covariant vectors $e^b{}_j$ related by the orthogonality condition

$$\begin{aligned} e^a{}_i e^i{}_b &= \delta^a{}_b, \\ e^i{}_a e^a{}_j &= \delta^i{}_j, \end{aligned} \tag{10}$$

and determining the metric tensor of that space

$$g_{ij} = \eta_{ab} e^a{}_i e^b{}_j, \;\; \eta_{ab} = diag(1\ -1\ -1\ -1). \tag{11}$$

In (10) and (11), i, j, k are world indices that run through the values $0, 1, 2, 3$, and a, b, c are local indices that are simply the numbers of a tetrad vector. We will agree that the local indices will also take on values $0, 1, 2, 3$, so that, for instance $e^2{}_0$ will denote the component in the second coordinate in the basic frame of the contravariant vector of the tetrad №0. The tetrads themselves can be regarded as reference frames in the space-time under consideration that meet some conditions. It follows from (11) that the space-time in the theory of physical vacuum is a stratified space, a layer being connected to the basic space of world coordinates x^i by the tetrads $e^i{}_a$ [5].

In fact, any vector $\mathbf{X}$, represented by the coordinates $\{X^i\}$ in the basic space, can be decomposed along the vectors $\mathbf{e}_a$ of the local basis

$$\mathbf{X} = \mathbf{e}_a X^a, \tag{12}$$

hence the local X^a and the basic X^i coordinates of the vector $\mathbf{X}$ are related by

$$X^a = e^a{}_i X^i. \tag{13}$$

The common symmetries of the space-time of physical vacuum are determined:

- first, by the transformations of coordinates describing the relative positions of the origins of arbitrary frames:

$$e^{i'}{}_a = \frac{\partial x^{i'}}{\partial x^i} e^i{}_a, \; \|\frac{\partial x^{i'}}{\partial x^i}\| \in T_4, \tag{14}$$

where T_4 is a translation group (a four-dimensional group of coordinate transformations) in the space of world coordinates (basis);

- second, by transformations of angular variables describing changes in mutual orientations (i.e., rotations) of arbitrary frames of reference:

$$e^i{}_{a'} = e^i{}_a \Lambda^a{}_{a'}, \; \Lambda^a{}_{a'} \in SO(1.3), \tag{15}$$

where $SO(1.3)$ is a pseudo-orthogonal group of rotations in a manifold of angular coordinates (layer).

An arbitrary vector $\mathbf{X}$ can thus be transformed using the world indices when transformed to another frame of reference:

$$X^{i'} = \frac{\partial x^{i'}}{\partial x^i} X^i, \; \|\frac{\partial x^{i'}}{\partial x^i}\| \in T_4, \tag{16}$$

and can be transformed using the local indices when the reference frame is turned (rotated):

$$X^{a'} = \Lambda^{a'}{}_a X^a, \; \Lambda^{a'}{}_a \in SO(1.3). \tag{17}$$

The motion of the center of a test particle in an absolute- parallelism space is given by [5]:

$$\frac{d^2x^i}{ds^2} + \Delta^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \tag{18}$$

where

$$\Delta^i_{jk} = e^i{}_a e^a{}_{j,k} \tag{19}$$

is the connectedness of the absolute-parallelism space, which can be represented as

$$\Delta^i_{jk} = \Gamma^i_{jk} + T^i_{jk}. \tag{20}$$

In (20), Γ^i_{jk} are Christoffel symbols, and T^i_{jk} are Ricci rotation coefficients. Because the field of Ricci rotation coefficients in the general case has a zero torsion $T^i_{jk} - T^i_{kj} \neq 0$, it is referred to as a torsion field.

When there are no external fields, the metric is flat, and in Cartesian coordinates $\Gamma^i_{jk} = 0$. Therefore, in the absence of external fields in Cartesian coordinates the equations of motion (18) can be written as:

$$\frac{du^i}{ds_0} + T^i_{jk}u^j u^k = 0. \tag{21}$$

We have here introduced the four-dimensional velocity of a particle in a given frame

$$u^i = \frac{dx^i}{ds_0} = \gamma_0(1,\, B_x,\, B_y,\, B_z), \qquad B_\alpha = \frac{V_\alpha}{c}, \tag{22}$$

where

$$\gamma_0 = \frac{1}{\sqrt{1-B^2}}, \qquad B^2 = B_x^2 + B_y^2 + B_z^2, \tag{23}$$

is a relativistic factor,

$$ds_0 = \sqrt{\eta_{ij}dx^i dx^j} = \frac{cdt}{\gamma_0},\ \eta_{ij} = diag(1\ -1\ -1\ -1) \tag{24}$$

is the interval of a plat space.

3. Ricci rotation coefficients (torsion field) in an inertial frame of reference.

In a space without external fields in an inertial frame of reference a free particle moves with a velocity that is constant in magnitude and direction $du^i/ds_0 = 0$. We thus have the condition

$$T^i{}_{jk} u^j u^k = 0. \tag{25}$$

This amounts to the requirement that the Ricci rotation coefficients be antisymmetric in two subscripts:

$$T^i{}_{jk} = -T^i{}_{kj}. \tag{26}$$

From the formula (19) in Cartesian coordinates we get the condition for the tetrad vectors

$$e^i{}_b e^b{}_{j,k} = -e^i{}_b e^b{}_{k,j}. \tag{27}$$

Multiplying both sides of the equation by $e^a{}_i$ and summing up over i, we obtain

$$e^a{}_{j,k} = -e^a{}_{k,j}. \tag{28}$$

The tetrad $e^a{}_i$ defines the metric tensor of the flat space

$$g_{ij} = \eta_{ij} = e^a{}_i \, \eta_{ab} \, e^b{}_j, \tag{29}$$

where

$$\eta_{ij} = \eta_{ab} = diag(1\ -1\ -1\ -1). \tag{30}$$

The conditions (28) and (29) enable the tetrad $e^a{}_i$ in Cartesian coordinates to be found as a trivial solution to

$$\begin{array}{l} e^a{}_i = \delta^a{}_i = diag(1\ \ 1\ \ 1\ \ 1), \\ e^j{}_b = \delta^j{}_b = diag(1\ \ 1\ \ 1\ \ 1). \end{array} \tag{31}$$

It follows from this that all the Ricci rotation coefficients and all the components of a nonholonomic object in Cartesian coordinates in an inertial reference frame are zero:

$$T^i{}_{jk} = e^i{}_a e^a{}_{j,k} = 0, \tag{32}$$

$$\Omega_{jk}^{\ \ ..i} = e^i{}_a e^a{}_{[k,j]} = 0. \tag{33}$$

In active and passive transformations over the world indices i, j, k (concerned with transitions to another inertial frame or to another coordinate system, respectively), the Ricci rotation coefficients $T^i{}_{jk}$ and the nonholonomic object

$\Omega_{jk}^{\cdot\cdot i}$ behave like tensors [5]. This suggests that: **in any inertial frame without external fields in any coordinates the Ricci rotation coefficients and the components of a nonholonomic object are zero**

$$T^i{}_{jk} = 0, \tag{34}$$

$$\Omega_{jk}^{\cdot\cdot i} = 0. \tag{35}$$

4. The orientation equation.

It will be recalled that the equation of motion (18) for a particle in an absolute-parallelism space describes the motion of the origin of the frame that is rigidly tied with the oriented point.

If formally applied, nonlinear coordinate transportations of general relativity, which allegedly correspond to a transition to a conventional noninertial reference frame, result in the following:

- against the background of a flat space (in the absence of external fields) some fictitious forces will emerge, which vary with the Christoffel symbols $\Gamma^{i'}_{j'k'}u^{j'}u^{k'}$, whose expressions may be at variance with classical mechanics [4] (for coordinate transportations corresponding to noncommuting rotations);

- forces that are proportional to $T^{i'}{}_{j'k'}$ will be equal to zero, since the Ricci rotation coefficients under any coordinate transportations transform as tensors

$$T^{i'}{}_{j'k'} = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{i'}}{\partial x^i}T^i{}_{jk} = 0. \tag{36}$$

Consequently, the equation of motion of a particle in world coordinates (18), strictly speaking, refers to an inertial frame of reference.

Therefore, a conventional noninertial frame of reference - an accelerated rotating frame - can be represented in the most complete form by its four-dimensional rotation in the manifold of angular variables:

$$e^i{}_{a'} = e^i{}_a\Lambda^a{}_{a'}, \;\; \Lambda^a{}_{a'} \in SO(1.3). \tag{37}$$

The primed local indices in (37) refer to the noninertial frame that performs a four-dimensional rotation about the initial inertial frame of reference.

Let us show that equation (18) exerts no influence on the intrinsic orientation of a traveling particle. Instead of (18) we will use the equivalent equation of motion for a particle in world coordinates:

$$u^i{}_{,k} + \Delta^i{}_{jk} u^j = 0, \tag{38}$$

where $,k$ is the partial derivative with respect to the contravariant coordinate x^k. A change to layer coordinates in (38) occurs using the tetrad $e^a{}_i$ помощью тетрады $e^a{}_i$

$$e^a{}_i u^i{}_{,k} + e^a{}_i \Delta^i{}_{jk} u^j = 0.$$

Hence we get the following equation:

$$\left(e^a{}_i u^i\right)_{,k} - e^a{}_{i,k} u^i + e^a{}_i \Delta^i{}_{jk} u^j = 0. \tag{39}$$

From the definition of the connectedness of absolute parallelism $\Delta^i_{jk} = e^i{}_a e^a{}_{j,k}$ we can easily get

$$e^a{}_{i,k} = e^a{}_l \Delta^l_{ik}. \tag{40}$$

Substituting (40) into (39) gives

$$u^a{}_{,k} - e^a{}_l \Delta^l_{ik} u^i + e^a{}_i \Delta^i{}_{jk} u^j = 0. \tag{41}$$

We note that the second and third terms in that equation cancel out to yield

$$u^a{}_{,k} = 0. \tag{42}$$

Multiplying both sides of equation (42) by $u^k = dx^k/ds$ and summing over k, we arrive at

$$\frac{du^a}{ds} = 0. \tag{43}$$

Condition (42) is a direct corollary of the equation of motion for the oriented particle (18), it was derived for the general case of $\Gamma^i{}_{jk} \neq 0$. We have thus shown that the general equation of motion does not affect the orientation of the oriented particle itself, but rather defines exclusively its motion "as a whole."

Accordingly, a body traveling along a world line in absolute- parallelism space must be described by a separate equation. This suggests that this equation must act in a manifold of angular variables and be invariant under transformations that make up the pseudo-orthogonal group $SO(1.3)$.

A candidate orientation equation can be the following equation:

$$\mathbf{u^a{}_{,k} - \Delta^a{}_{bk} u^b = 0}. \tag{44}$$

We will now show that the orientation equation (44) follows from the most general properties of absolute-parallelism space. In (44) $\Delta^a{}_{bk} = e^a{}_i \Delta^i{}_{jk} e^j{}_b$, which by the definition of the connectedness of absolute parallelism (19), can be transformed as follows:

$$\Delta^a{}_{bk} = e^a{}_{l,k} e^l{}_b = -e^a{}_l e^l{}_{b,k}. \tag{45}$$

We multiply both sides of (45) by $e^b{}_m$ and sum over b to get

$$\Delta^a{}_{bk} e^b{}_m = e^a{}_{l,k} e^l{}_b e^b{}_m = e^a{}_{l,k} \delta^l_m. \tag{46}$$

Hence

$$e^a{}_{m,k} - \Delta^a{}_{bk} e^b{}_m = 0. \tag{47}$$

Multiplying each term of the equation by the four-dimensional velocity $u^k = dx^k/ds$ and summing over k, we have

$$\frac{de^a{}_m}{ds} - \Delta^a{}_{bk} e^b{}_m u^k = 0. \tag{48}$$

Equations (48) are a set of six equations for six independent components of the matrix $e^a{}_m$. On the other hand, equations (48) can be referred to as a relativistic generalization of the Frenet formulas for an absolute-parallelism space.

These equations take on a simpler form in the accompanying frame of reference, the one in which the particle is at rest, and the four-dimensional velocity vector is

$$u^k = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{49}$$

For the accompanying frame, equations (48) can we written as

$$\frac{de^a{}_m}{ds} - \Delta^a{}_{b0} e^b{}_m = 0. \tag{50}$$

Equation (50) is a Frenet equation for an absolute-parallelism space, which were initially formulated in an accompanying frame.

Note that the orientation equations (44) are a set of four equations for the four local components of the velocity u^a. The orientation equations can be obtained from (47) by subjecting them to the additional condition that $e^a{}_0 = u^a$

$$u^a{}_{,k} - \Delta^a{}_{bk} u^b = 0. \tag{51}$$

To prove the invariance of (44) under a group of transformations $SO(1.3)$ we will derive the law of the transformation $\Delta^{a'}_{b'k} \rightarrow \Delta^{a''}_{b''k}$ under a four-dimensional rotation of the frame of reference (37). From (45), we get

$$\begin{aligned} \Delta^{a''}_{b''k} &= -\, e^{a''}_{l}\, e^{l}{}_{b'',k} = -\,(\Lambda^{a''}_{a'}\, e^{a'}_{l})\,(e^{l}{}_{b'}\, \Lambda^{b'}_{b''})_{,k} = \\ &= \Lambda^{a''}_{a'}\,(-e^{a'}_{l}\, e^{l}{}_{b',k})\,\Lambda^{b'}_{b''} - \Lambda^{a''}_{a'}\, e^{a'}_{l}\, e^{l}{}_{b'}\,\Lambda^{b'}_{b'',k} = \\ &= \Lambda^{a''}_{a'}\,\Delta^{a'}_{b'k}\,\Lambda^{b'}_{b''} - \Lambda^{a''}_{a'}\,\Lambda^{a'}_{b'',k}, \end{aligned} \tag{52}$$

where $\Lambda^{a''}_{a'} \in SO(1.3)$ are the matrices of transition from one rotating frame to another rotating frame. Equation (52) shows that the quantity $\Delta^a{}_{bk}$ in a layer is transformed not as connectedness in a vector space (in that case there would be a + sign in the last line in (52)), but in another manner. We will dub it "pseudo-connectedness of absolute parallelism in a layer"because $\Delta^a{}_{bk}$ gets transformed in a peculiar manner.

We will also derive the transformation law $T^{a'}_{b'k} \rightarrow T^{a''}_{b''k}$. In the general case, the quantities T^a_{bk} are expressed in terms of the components of tetrad vectors [5]:

$$T^a_{bk} = \left(\nabla_k\, e^a{}_j\right)\, e^j{}_b,$$

where ∇_k is a covariant derivative with respect to the connectedness of the Riemannian space Γ^i_{jk}. Therefore,

$$\begin{aligned} T^{a''}_{b''k} &= \left(\nabla_k\, \Lambda^{a''}_{a'}\, e^{a'}_{j}\right)\, e^j{}_{b'}\Lambda^{b'}_{b''} = \\ &= \left[\left(\nabla_k\, \Lambda^{a''}_{a'}\right)\, e^{a'}_{j} + \Lambda^{a''}_{a'}\left(\nabla_k\, e^{a'}_{j}\right)\right]\, e^j{}_{b'}\Lambda^{b'}_{b''} = \\ &= \left(\nabla_k\, \Lambda^{a''}_{a'}\right)\, \Lambda^{a'}_{b''} + \Lambda^{a''}_{a'}\, T^{a'}_{b'k}\, \Lambda^{b'}_{b''}. \end{aligned} \tag{53}$$

When there are no external fields $\Gamma^i_{jk} = 0$, formula (53) can be simplified as follows:

$$T^{a''}_{b''k} = \Lambda^{a''}_{a'}\, T^{a'}_{b'k}\, \Lambda^{b'}_{b''} + \Lambda^{a''}_{a',k}\, \Lambda^{a'}_{b''}. \tag{54}$$

Noting that $\Lambda^{a''}_{a',k}\, \Lambda^{a'}_{b''} = -\Lambda^{a''}_{a'}\, \Lambda^{a'}_{b'',k}$, we can also write (54) as

$$T^{a''}_{b''k} = \Lambda^{a''}_{a'}\, T^{a'}_{b'k}\, \Lambda^{b'}_{b''} - \Lambda^{a''}_{a'}\, \Lambda^{a'}_{b'',k}. \tag{55}$$

This relationship indicates that in the absence of external fields $T^a{}_{bk}$ and $\Delta^a{}_{bk}$ are transformed similarly, and hence they coincide.

Let us show that the orientation equation is independent of the choice of a reference frame. To this end, we write the orientation equation in an arbitrarily rotating frame

$$u^{b''}{}_{,k} - \Delta^{b''}{}_{a''k} u^{a''} = 0 \tag{56}$$

and show that in another rotating four-dimensional frame it will have the same form.

If we now substitute into (56) the formula (52) and the relationship for the local components of the four-dimensional velocity of a particle in various frames of reference

$$u^{b''} = \Lambda^{b''}{}_{b'} u^{b'}, \ \Lambda^{b''}{}_{b'} \in SO(1.3),$$

we will obtain

$$\Lambda^{b''}{}_{b',k}\, u^{b'} + \Lambda^{b''}{}_{b'}\, u^{b'}{}_{,k} \Lambda^{b''}{}_{b'}\, \Delta^{b'}{}_{a'k}\, \Lambda^{a'}{}_{a''}\, \Lambda^{a''}{}_{c'}\, u^{c'} + \Lambda^{b''}{}_{a'}\, \Lambda^{a'}{}_{a'',k}\, \Lambda^{a''}{}_{c'}\, u^{c'} = 0. \tag{57}$$

Multiplying both sides of (57) by $\Lambda^{d'}{}_{b''}$ and summing over b'', we get

$$\Lambda^{d'}{}_{b''}\, \Lambda^{b''}{}_{b',k}\, u^{b'} + u^{d'}{}_{,k} - \Delta^{d'}{}_{c'k}\, u^{c'} + \Lambda^{d'}{}_{a'',k}\, \Lambda^{a''}{}_{c'}\, u^{c'} = 0. \tag{58}$$

We note that the first and last terms of (58) cancel out due to the relationship

$$\Lambda^{d'}{}_{b''}\, \Lambda^{b''}{}_{b',k} + \Lambda^{d'}{}_{a'',k}\, \Lambda^{a''}{}_{b'} = 0.$$

We will thus arrive at the equation

$$u^{d'}{}_{,k} - \Delta^{d'}{}_{c'k} u^{c'} = 0, \tag{59}$$

which is the orientation equation for an oriented point in another frame of reference. We have thus shown that the form of the orientation equation is independent of the choice of a four- dimensional rotating frame. In other words, the orientation equation is invariant under the transformation group $SO(1.3)$. In a similar manner, we can prove the invariance under transformations $SO(1.3)$ of the matrix equations (48).

Now show that the proposed equation of orientation does not contain direct information about the motion of a particle "as a whole", i.e., it does not exert any direct influence on the motion of the origin of the frame associated with the oriented point. To this end, in the orientation equation we will change over to the basis coordinates. Substituting into (44) the relationships

$$u^b = e^b{}_m\, u^m, \;\; u^a{}_{,k} = e^a{}_{m,k}\, u^m + e^a{}_m\, u^m{}_{,k},$$

and (45), yields

$$e^a{}_{m,k}\, u^m + e^a{}_m\, u^m{}_{,k} + e^a{}_m\, e^m{}_{b,k}\, e^b{}_n u^n = 0. \qquad (60)$$

Multiplying both sides of (60) by $e^p{}_a$ and summing over a, we get

$$e^p{}_a\, e^a{}_{m,k}\, u^m + \delta^p_m\, u^m{}_{,k} + e^p{}_{b,k}\, e^b{}_n\, u^n = 0.$$

Since the first and third terms of the equation cancel out, by (45), we arrive at

$$u^p{}_{,k} = 0, \qquad (61)$$

which goes to prove our supposition that there is no direct influence of the orientation equation of the motion of a particle "as a whole".

Nevertheless, the equation of motion of the particle (18) and its orientation equation (44) are not independent of each other. They are related via various components of the common quantity Δ: $\Delta^i{}_{jk}$ и $\Delta^a{}_{bk}$.

Therefore, the motion of an oriented point in an absolute- parallelism space can only be described completely by solving a set of ten equations - of the four equations of motion of the particle and of the six equations for the component of the benchmark $e^a{}_m$

$$\begin{aligned} \frac{du^i}{ds} + \Delta^i{}_{jk}\, u^j\, u^k &= 0, \\ \frac{de^a{}_m}{ds} - \Delta^a{}_{bk} e^b{}_m u^k &= 0. \end{aligned} \qquad (62)$$

The set of equations (62) defines the four world coordinates of a particle x^i, $i = 0, 1, 2, 3$ and the six independent components of the intrinsic benchmark of the oriented particle $e^a{}_i$, thus uniquely defining the six angles of the orientation of the particle.

The complete set of equations of motion of a particle in an absolute-parallelism space can be written in terms of the components of the local velocity of a particle (m is the particle's mass)

$$m\,\frac{du^i}{ds} + m\,\Delta^i{}_{jk}\,u^j\,u^k = 0,$$
$$m\,\frac{du^a}{ds} - m\,\Delta^a{}_{bk}\,u^b\,u^k = 0. \qquad (63)$$

The set (63) consists of the four equations of motion of a particle and the four equations for the local velocity ("orientation equations") for the eight unknowns - the four world coordinates (or four components of the velocity of the particle) and the four components of the local velocity of the particle. It is to be noted that these quantities will also completely define the motion of the oriented particle in an absolute-parallelism space, since the two missing parameters (out of ten) are specified by the well-known calibration conditions $e^a{}_0 = u^a$ и $e^i{}_0 = u^i$.(The second calibration condition was used to derive the equations of motion (18) [5]).

In the first equation of the set, the equation of motion for the oriented particle, which, strictly speaking, refers to an inertial frame of reference, the main action on the particle is exerted by an external field, since actually

$$\Delta^i{}_{jk}\,u^j\,u^k = \Gamma^i{}_{jk}\,u^j\,u^k. \qquad (64)$$

In the second equation of the set, the equation of orientation of the oriented point, the main contribution is made from the Ricci rotation coefficients in layer coordinates, since [7]

$$\Delta^a{}_{bc} = T^a_{bc}. \qquad (65)$$

In terms of the layers coordinates, the force is only determined by the torsion properties of an absolute-parallelism geometry.

The pseudo-connectedness of absolute parallelism $\Delta^a{}_{bc}$ being predominantly influenced by T^a_{bc}, we can write the orientation equation (44) completely in layer variables as follows:

$$m\,\frac{du^a}{ds} - m\,T^a_{bc}\,u^b\,u^c = 0, \qquad (66)$$

and we will refer to it as **the torsion equation.**

One of the sources of the torsion field T^a_{bk} that determines the orientation properties of an oriented point can be the torsion component of an external field. In fact, any external field is characterized by a set of components of the Christoffel symbols $\Gamma^i{}_{jk}$ and of the components of the Ricci rotation coefficients T^a_{bk} [5]. Here a purely torsion external field is possible that refers to the inertial frame of reference

$$\begin{aligned} \Gamma^i{}_{jk} &= 0, \\ T^a_{bk} &\neq 0, \end{aligned} \tag{67}$$

which affects the four-dimensional orientation of the oriented point.

We note that the equation of motion and the orientation equation determine, generally speaking, the components of one quantity that is concerned with the oriented point, namely the world (u^i) and local (u^a) components of the four-dimensional velocity, which are rigidly linked to one another

$$\begin{aligned} u^i &= e^i{}_a\, u^a, \\ u^b &= e^b{}_j\, u^j. \end{aligned} \tag{68}$$

It is not to be excluded that the torsion components of an external field T^a_{bk} may bring about such a change in the local velocity of the particle u^a that will entail a change in the world components u^i of the particle velocity. It follows that under certain conditions **real forces can emerge that are only due to the torsion component of the external field.**

5. Transition to a noninertial frame of reference.

We will demonstrate that another source of a torsion field can be the space-time itself relative to a frame involved in a four- dimensional rotation, and that the forces engendered in the process can be interpreted only as inertial forces that are well known in classical mechanics.

We will confine ourselves to the case of the absence of external fields $\Gamma^i{}_{jk} = 0$. Since all the components of the Ricci rotation coefficients in an inertial frame of reference in that case vanish, then also so do the Ricci rotation coefficients expressed in terms of the layer variables

$$T^a_{bk} = 0. \tag{69}$$

Furthermore, in relation to the inertial frame, in the absence of external fields

$$\Delta^a{}_{bk} = T^a_{bk} = 0. \tag{70}$$

Therefore, the set of equations (63) in an inertial frame without external fields becomes

$$\begin{aligned} m\tfrac{du^i}{ds} &= 0, \\ m\tfrac{du^a}{ds} &= 0. \end{aligned} \tag{71}$$

If we then pass over from an inertial, i.e., nonrotating frame, to a rotating four-dimensional frame of reference, formula (52) simplifies to

$$\Delta^{a'}{}_{b'k} = \Lambda^{a'}{}_{a}\,\Delta^{a}{}_{bk}\,\Lambda^{b}{}_{b'} \,-\, \Lambda^{a'}{}_{a}\,\Lambda^{a}{}_{b',k} \,=-\,\Lambda^{a'}{}_{a}\,\Lambda^{a}{}_{b',k}. \tag{72}$$

The orientation equation for a particle relative to a frame involved in a four-dimensional rotation can thus be written as

$$m\,\frac{du^{a'}}{ds} \,+\, m\,\Lambda^{a'}{}_{a}\,\Lambda^{a}{}_{b',k}\,u^{b'}\,u^{k} = 0, \tag{73}$$

where $\Lambda^{a'}{}_{a} \in SO(1.3)$ are matrices of transition from an inertial (nonrotating) frame to a rotating frame of reference. We note that in (73) are the components of a four- dimensional velocity of a particle in the initial nonrotating frame.

Consider an initial inertial frame of reference, such that in it a particle would not move relative to it and would reside at its origin. Thus,

$$u^k \;=\; \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{74}$$

and (73) reduces to a simpler expression

$$m\,\frac{du^{a'}}{ds} \;=-m\,\Lambda^{a'}{}_{a}\,\Lambda^{a}{}_{b',0}\,u^{b'}. \tag{75}$$

This equation is written in the layer coordinates of the frame performing a four-dimensional rotation in the initial inertial frame of reference. The right side of the equation contains a counterpart of force expressed in terms of the local layer coordinates

$$F^{a'} \;=\; -m\,\Lambda^{a'}{}_{a}\,\Lambda^{a}{}_{b',0}\,u^{b'} \;=\; m\,\Lambda^{a'}{}_{a,0}\,\Lambda^{a}{}_{b'}\,u^{b'}. \tag{76}$$

This suggests that solving (75), we will obtain the particle path in terms of the local layer coordinates. To compare with experimental data we will have to project this path on the vector space of the world coordinates (basis), where traditionally velocities and accelerations of particles are measured, and to study the projection obtained.

However, for small specific forces $F^{a'}/m$ engendered in the four-dimensional rotation of a noninertial frame of reference

$$\frac{F^{a'}}{m} = \Lambda^{a'}{}_{a,0}\,\Lambda^{a}{}_{b'}\,u^{b'} \ll 1, \tag{77}$$

we can make an assumption that drastically simplifies our investigation. Namely, if the conditions (77) are met, we will assume that the projection of the true path of a particle onto the world coordinate space will not differ from the particle path derived by solving the equation of motion of a particle, in which the right side contains the projection of the force onto the world coordinate space

$$F^{i'} = e^{i'}{}_{a'}\,F^{a'}. \tag{78}$$

The sought-for equation of motion of a particle in a noninertial frame in terms of world coordinates will thus be

$$m\,\frac{du^{i'}}{ds} = F^{i'}, \tag{79}$$

where

$$\begin{aligned} F^{i'} &= e^{i'}{}_{a'}\,F^{a'} = e^{i'}{}_{a'}\,m\,\Lambda^{a'}{}_{a,0}\,\Lambda^{a}{}_{b'}\,u^{b'} = \\ &= m\,e^{i'}{}_{a}\,\Lambda^{a}{}_{a'}\,\Lambda^{a'}{}_{a,0}\,\Lambda^{a}{}_{b'}\,\Lambda^{b'}{}_{b}\,e^{b}{}_{j'}\,u^{j'} = \\ &= m\,e^{i'}{}_{a}\,\Lambda^{a}{}_{a'}\,\Lambda^{a'}{}_{b,0}\,e^{b}{}_{j'}\,u^{j'}. \end{aligned} \tag{80}$$

Since in passing to a frame performing a four-dimensional rotation (37) no transformation in world indices $i \rightarrow i'$ was carried out, the vectors of the tetrad $e^{i'}{}_{a}$ remained unchanged compared with $e^{i}{}_{a}$, i.e.,

$$\begin{aligned} e^{i'}{}_{a} &= \delta^{i'}_{a}, \\ e^{b}{}_{j'} &= \delta^{b}_{j'}, \end{aligned} \tag{81}$$

and the expression for the force $F^{i'}$ can be written as follows

$$F^{i'} = m\,P^{i'}{}_{j'}\,u^{j'}, \quad P^{i'}{}_{j'} = \delta^{i'}_{a}\,\Lambda^{a}{}_{a'}\,\Lambda^{a'}{}_{b,0}\,\delta^{b}_{j'}. \tag{82}$$

In (82), $\Lambda^{a'}_{\ a}$ are the matrices of rotation in a pseudo-Euclidian space, which correspond to a transformation from an inertial frame in which the oriented point is at rest, to a noninertial frame performing an arbitrary four-dimensional rotation.

The most general matrix of spatial rotation in a pseudo-Euclidean space looks like

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_{xx} & \cos\theta_{xy} & \cos\theta_{xz} \\ 0 & \cos\theta_{yx} & \cos\theta_{yy} & \cos\theta_{yz} \\ 0 & \cos\theta_{zx} & \cos\theta_{zy} & \cos\theta_{zz} \end{pmatrix}, \tag{83}$$

where $\cos\theta_{\alpha\beta}$ are direction cosines of the angles [8].

The most general matrix of rotation in a pseudo- Euclidean plane not concerned with a spatial rotation looks like [8]:

$$L = \begin{pmatrix} \gamma & -\beta_x\gamma & -\beta_y\gamma & -\beta_z\gamma \\ -\beta_x\gamma & 1+\frac{(\gamma-1)\beta_x^2}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} \\ -\beta_y\gamma & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & 1+\frac{(\gamma-1)\beta_y^2}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} \\ -\beta_z\gamma & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & 1+\frac{(\gamma-1)\beta_z^2}{\beta^2} \end{pmatrix}, \tag{84}$$

where $\beta_\alpha = v_\alpha/c$, v_α are the components of the velocity of the origin of the traveling frame in relation to the rest one,

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2 \tag{85}$$

is a relativistic factor.

In the general case, the frame's four-dimensional rotation that gives rise to a noninertial frame can be expanded into the rotation in a pseudo-Euclidean plane and the purely spatial rotation. It is common knowledge that the finite rotations L and R in the general case do not commute and that the product L and R is determined by their sequence.

Therefore, the matrix of four-dimensional rotation can be written as

$$\Lambda^{a''}_{\ a} = R^{a''}_{\ a'}\, L^{a'}_{\ a}, \tag{86}$$

where $R^{a''}_{\ a'}$ and $L^{a'}_{\ a}$ are the respective rotation matrices with time-dependent parameters, or as

$$\Lambda^{a''}_{\ a} = L^{a''}_{\ a'}\, R^{a'}_{\ a}. \tag{87}$$

We will associate the first case with the noninertial frame of the first kind; and the second, to a noninertial frame of the second kind. We will show that forces acting on a particle in the two frames in question are different, and will interpret the results obtained.

Substituting (86) and (87) into (82) and using the following properties of quadratic matrices

$$(A\,B)^{-1} = B^{-1}\,A^{-1}, \qquad A^{-1}\,A = I,$$

we will derive formulas for $P^{i'}_{\ j'}$ for the two cases, expressed in terms of parameters of purely spatial rotation and of rotation in a pseudo-Euclidean plane

$$\begin{aligned} P^{i'}_{\ j'}{}^{(RL)} &= e^{i'}_{\ a}\left(\Lambda^{a''}_{\ a}\right)^{-1}\Lambda^{a''}_{\ b,0}\,e^{b}_{\ j'} = \\ &= \delta^{i'}_{a}\left(R^{a''}_{\ a'}\,L^{a'}_{\ a}\right)^{-1}\left(R^{a''}_{\ a'}\,L^{a'}_{\ b}\right)_{,0}\,\delta^{b}_{j'} = \\ &= \delta^{i'}_{a}\left[\left(L^{a'}_{\ a}\right)^{-1}\left(R^{a''}_{\ a'}\right)^{-1}R^{a''}_{\ a',0}\,L^{a'}_{\ b} + \left(L^{a'}_{\ a}\right)^{-1}L^{a'}_{\ b,0}\right]\delta^{b}_{j'}, \end{aligned} \tag{88}$$

for a noninertial frame of the first kind, and

$$P^{i'}_{\ j'}{}^{(LR)} = \delta^{i'}_{a}\left[\left(R^{a'}_{\ a}\right)^{-1}\left(L^{a''}_{\ a'}\right)^{-1}L^{a''}_{\ a',0}\,R^{a'}_{\ b} + \left(R^{a'}_{\ a}\right)^{-1}R^{a'}_{\ b,0}\right]\delta^{b}_{j'}, \tag{89}$$

for a noninertial frame of the second kind.

6. Nonrelativistic approximation.

This section addresses the nonrelativistic case, where the velocity of the origin of a noninertial frame in relation to an inertial frame is small as compared with the velocity of light $\beta \ll 1$. We will confine ourselves to a case where the origin of a noninertial frame of reference moves in one plane $XOY \quad (\beta_z = \dot{\beta}_z = 0)$ and the axis of spatial rotation of the frame is normal to that plane

$$\mathbf{\Omega} = (0,\,0,\,\Omega) = (0,\,0,\,\frac{d\varphi}{dt}). \tag{90}$$

In that case, the matrix of spatial rotation is given by:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{91}$$

where $\varphi = \varphi(t)$ is the angle of rotation about the z axis.

The matrix of pseudo-Euclidean rotation, to within $o(\beta)$, is expressed as follows:

$$L = \begin{pmatrix} 1 & -\beta_x\gamma & -\beta_y\gamma & 0 \\ -\beta_x\gamma & 1 & 0 & 0 \\ -\beta_y\gamma & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{92}$$

where $\beta_x = \beta_x(t), \quad \beta_y = \beta_y(t), \quad \beta^2 = \beta^2(t) = \beta_x^2(t) + \beta_y^2(t)$.

Without loss of generality, we will assume that at a time t the origins of the noninertial and inertial frames coincide.

Direct computation of the matrix elements $P^{i'}_{\ j'}$ to within $o(\beta)$ gives

$$P^{i'}_{\ j'}{}^{(RL)} = \frac{1}{c}\begin{pmatrix} 0 & -\dot\beta_x - \beta_y\Omega & -\dot\beta_y + \beta_x\Omega & 0 \\ -\dot\beta_x - \beta_y\Omega & 0 & \Omega & 0 \\ -\dot\beta_y + \beta_x\Omega & -\Omega & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{93}$$

$$P^{i'}_{\ j'}{}^{(LR)} = \frac{1}{c}\begin{pmatrix} 0 & -\dot\beta_x\cos\varphi + \dot\beta_y\sin\varphi & -\dot\beta_y\cos\varphi - \dot\beta_x\sin\varphi & 0 \\ -\dot\beta_x\cos\varphi + \dot\beta_y\sin\varphi & 0 & \Omega & 0 \\ -\dot\beta_y\cos\varphi - \dot\beta_x\sin\varphi & -\Omega & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{94}$$

From (79), we will write the equation of motion for a particle - the origin O of an inertial frame of reference - in a noninertial frame of reference:

$$\frac{\gamma'}{c}\frac{du^{i'}}{dt} = P^{i'}_{\ j'}u^{j'}, \tag{95}$$

or in matrix form:

$$\frac{\gamma'}{c}\frac{d}{dt}\begin{pmatrix} \gamma' \\ \gamma' B'_x \\ \gamma' B'_y \\ \gamma' B'_z \end{pmatrix} = P^{i'}_{\ j'}\begin{pmatrix} \gamma' \\ \gamma' B'_x \\ \gamma' B'_y \\ 0 \end{pmatrix}. \tag{96}$$

It is to be remembered that the velocity $\mathbf{V}'$ of point relative to a noninertial frame of reference is equal in magnitude and opposite in direction to the velocity

$\mathbf{v}$ of the origin of the noninertial frame relative to the inertial frame:

$$\begin{aligned} V'_\alpha &= -v_\alpha, \\ B'_\alpha &= V'_\alpha/c = -\beta_\alpha. \end{aligned} \tag{97}$$

Therefore, the relativistic factor of a particle γ' in a noninertial frame is

$$\gamma' = \frac{1}{\sqrt{1 - B'^2}} = \frac{1}{\sqrt{1-\beta^2}} = \gamma. \tag{98}$$

7. Noninertial frame of reference of the first kind.

From (93) and (96) we can derive the four equations of motion for a body in a noninertial frame of the first kind in the nonrelativistic case to within $o(\beta)$.

Equation for $i' = 0$:

$$\frac{d\gamma'}{dt} = -\dot{\beta}_x B'_x - \dot{\beta}_y B'_y. \tag{99}$$

Multiplying both side of this equation by mc^2 and considering that $E = \gamma' mc^2$ is the complete energy of the body, we will obtain the equation for the power of the inertial forces:

$$\frac{dE}{dt} = -mw_x V'_x - mw_y V'_y, \tag{100}$$

where $w_\alpha = \dot{\beta}_\alpha c$ is the acceleration of the origin of a noninertial frame of reference. Note that, by (100), the work is done by forces due to the acceleration of the origin of a noninertial frame of reference.

Equation for $i' = 1$:

$$\frac{dB'_x}{dt} = -\dot{\beta}_x - \beta_y \Omega + B'_y \Omega = -\dot{\beta}_x + 2B'_y \Omega. \tag{101}$$

Equation for $i' = 2$:

$$\frac{dB'_y}{dt} = -\dot{\beta}_y + \beta_x \Omega - B'_x \Omega = -\dot{\beta}_y - 2B'_x \Omega. \tag{102}$$

Equation for $i' = 3$:

$$\frac{dB'_z}{dt} = 0. \tag{103}$$

We have used here $\beta_\alpha = -B'_\alpha$. Multiplying the left and right sides of these equations by mc, we will get the equations of motion for a particle in a noninertial frame of the first kind in the nonrelativistic case:

$$m\frac{dV'_x}{dt} = -mw_x + 2mV'_y\Omega, \tag{104}$$

$$m\frac{dV'_y}{dt} = -mw_y - 2mV'_x\Omega, \tag{105}$$

$$m\frac{dV'_z}{dt} = 0. \tag{106}$$

If we now write (104)-(106) in vector form

$$m\frac{d\mathbf{V}'}{dt} = -m\mathbf{w} + 2m[\mathbf{V}'\boldsymbol{\Omega}], \tag{107}$$

we will see that these equations are identical to the classical equations of motion of a particle in a noninertial frame of reference provided that at a time t the particle passed through the origin of that reference frame [4].

In 1921 Einstein in his work "The meaning of relativity"[2] noted that ": the inertia law, obviously, makes us to assign to a space-time continuum some objective properties. ... In order to elaborate on this idea within the framework of the present-day theory of action via a medium the properties of the space-time continuum that determine inertia must be regarded as field properties of space similar to an electromagnetic field."

The founder of the theory of physical vacuum G. Shipov, as early as 1979 [7], called the field of the Ricci rotation coefficients precisely an inertial field and stressed the identity of the field $T^i{}_{jk}$ and the inertial field. In his book "A Theory of Physical Vacuum", Shipov maintained that "the inertial field $T^i{}_{jk}$ is generated by the four-dimensional rotation of a reference frame, so that in the translational group T_4 these fields transform as a tensor, and in the rotation group $O(3.1)$ they transform according to a nontensor law."

This work, in accordance with the program of universal relativity theory [5], contains a consistent proof of the fact that a torsion field generated by the very space-time relative to a noninertial frame of the first kind engenders forces that are inertial forces.

8. Noninetial referece frame of the second kind.

In order to derive equations of motion for an oriented particle in a noninertial reference frame of the second kind, we note that any transition from one initial frame to another frame implies actually a rotation of the initial frame through a finite pseudo- Euclidean angle $\Delta\theta$ and through a finite three- dimensional angle $\Delta\varphi$. There can be an infinite number of paths resulting in such a finite turn in a manifold of angular variables. However, at each point on such a path (locally) a reference frame performing a four-dimensional rotation can be implemented as a noninertial frame of the first or second kind.

Therefore, what kind will a noninertial reference frame have will be conditioned by its local parameters with reference to the general path of rotation, namely by its local pseudo-Euclidean rotation $d\theta$ and the local three-dimensional rotation $d\varphi$ in a given point of the path.

This suggests that in a formula for a noninertial frame of the second kind, the angle φ must be taken to be small. This will yield the following equations of motion for an oriented particle:

Equation for $i' = 0$:

$$\frac{d\gamma'}{dt} = -\dot{\beta}_x B'_x - \dot{\beta}_y B'_y. \tag{108}$$

Equation for $i' = 1$:

$$\frac{dB'_x}{dt} = -\dot{\beta}_x + \dot{\beta}_y\varphi + B'_y\Omega = -\dot{\beta}_x + B'_y\Omega. \tag{109}$$

Equation for $i' = 2$:

$$\frac{dB'_y}{dt} = -\dot{\beta}_y - \dot{\beta}_x\varphi - B'_x\Omega = -\dot{\beta}_y - B'_x\Omega. \tag{110}$$

Equation for $i' = 3$:

$$\frac{dB'_z}{dt} = 0. \tag{111}$$

In (109) and 110) we ignored the small quantity of the second order $\dot{\beta}_\alpha\varphi$.

We now write equations for $i' = 1, 2, 3$ in vector form

$$m\frac{d\mathbf{V}'}{dt} = -m\mathbf{w} + m[\mathbf{V}'\boldsymbol{\Omega}]. \tag{112}$$

We see that the expression for the force due to intrinsic rotation of a noninertial reference frame of the second kind differs from the expression for the classical Coriolis force by the numerical coefficient 1/2.

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