# ABSOLUTE PARALLELISM GEOMETRY, RICCI AND CARTAN TORSIONS

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## 1 Object of anholonomicity -Ricci torsion. Connection of absolute parallelism

Consider a four-dimensional differentiable manifold with coordinates $x^i$ $(i = 0, 1, 2, 3)$ such that at each point of the manifold we have a vector $e^a{}_i$ $(i = 0, 1, 2, 3)$ and a covector $e^j{}_b$ $(b = 0, 1, 2, 3)$ with the normalization conditions

$$e^a{}_i e^j{}_a = \delta^j_i, \quad e^a{}_i e^i{}_b = \delta^a_b. \tag{1}$$

For arbitrary coordinate transformations

$$dx^{i'} = \frac{\partial x^{i'}}{\partial x^k} dx^k \tag{2}$$

in coordinate index $i$ the tetrad $e^a{}_i$ transforms as a vector

$$e^a{}_{i'} = \frac{\partial x^i}{\partial x^{i'}} e^a{}_i. \tag{3}$$

In the process, in the tetrad index $a$ relative to the transformations (2) it behaves as a scalar.

Tetrad $e^a{}_i$ defines the metric tensor of a space of absolute parallelism

$$g_{ik} = \eta_{ab} e^a{}_i e^b_k, \eta_{ab} = \eta^{ab} = \mathrm{diag}(1\ -1\ -1\ -1) \tag{4}$$

and the Riemannian metric

$$ds^2 = g_{ik} dx^i dx^k. \tag{5}$$

Using the tensor (4) and the normal rule, we can construct the Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m}). \tag{6}$$

that transform following a nontensor law of transformation [1]

$$\Gamma^{k'}_{j'i'} = \frac{\partial^2 x^k}{\partial x^{i'} \partial x^{j'}} \frac{\partial x^{k'}}{\partial x^k} + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^{k'}}{\partial x^k} \Gamma^k_{ji} \tag{7}$$

with respect to the coordinate transformations (2). In the relationship (6) and farther on we will denote the partial derivative with respect to the coordinates $x^i$ as

$$, k = \frac{\partial}{\partial x_k}. \tag{8}$$

Differentiating the arbitrary vector $e^a{}_i$ gives

$$e^a{}_{i,j'} = \frac{\partial x^j}{\partial x^{j'}} e^a{}_{i,j}. \tag{9}$$

Applying the differentiation operation (9) to the relationship (3) gives

$$e^a_{i',j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} e^a{}_{i,j} + \frac{\partial^2 x^i}{\partial x^{i'} \partial x^{j'}} e^a{}_i. \tag{10}$$

Alternating the indices $i'$ and $j'$ and subtracting from (10) the resultant expression, we have

$$e^a{}_{i',j'} - e^a{}_{j',i'} = (e^a{}_{i,j} - e^a{}_{j,i}) \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}}.$$

Considering (3), we can rewrite this relationship in the form

$$e^{k'}{}_a (e^a{}_{i',j'} - e^a{}_{j',i'}) = e^k{}_a (e^a{}_{i,j} - e^a{}_{j,i}) \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^{k'}}{\partial x^k}.$$

By definition, the differential

$$ds^a = e^a{}_i dx^i \tag{11}$$

is said to be complete, if the following relationship holds:

$$e^a{}_{i,j} - e^a{}_{j,i} = 0. \tag{12}$$

Otherwise, for $e^a{}_{i,j} - e^a{}_{j,i} \neq 0$, the differential (11) is not integrable (equality (12) is the condition of integration for the relationship (11)).

We will introduce the following geometric object [2]

$$\Omega^{..i}_{jk} = e^i{}_a e^a_{[k,j]} = \frac{1}{2} e^i{}_a (e^a{}_{k,j} - e^a{}_{j,k}) \tag{13}$$

with a tensor law of transformation relative to the coordinate transformations (2)

$$\Omega^{..i'}_{j'k'} = \Omega^{..i}_{jk} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^{i'}}{\partial x^i}. \tag{14}$$

Clearly, if the condition (12) is met, this object vanishes. In that case, tetrad $e^a{}_i$ is holonomic and the metric (5) characterizes holonomic differential geometry. If the object (13) is nonzero, we deal with *anholonomic* differential geometry, and the object (13) itself is called an object of *anholonomicity.*

We will rewrite the relationship (10) in the following manner:

$$\begin{aligned} e^a_{i',j'} &= \frac{\partial^2 x^i}{\partial x^{i'} \partial x^{j'}} e^a{}_i + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} e^a{}_{i,j} = \\ &= \left( \frac{\partial^2 x^k}{\partial x^{i'} \partial x^{j'}} + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \Delta^k_{ij} \right) e^a{}_k, \end{aligned} \tag{15}$$

where we have introduced the notation

$$\Delta^k_{ij} = e^k{}_a e^a{}_{i,j} \tag{16}$$

and used the orthogonality condition (1).

It is seen from the relationships (15 ) that the object $\Delta^k_{ij}$ gets transformed relative to the transformations (2) as the connection

$$\Delta^{k'}_{i'j'} = \frac{\partial^2 x^k}{\partial x^{i'} \partial x^{j'}} \frac{\partial x^{k'}}{\partial x^k} + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^{k'}}{\partial x^k} \Delta^k_{ij}. \qquad (17)$$

The connection of a space given by (16) is called the connection of absolute parallelism [3].

Interchanging in (17) the indices $i$ and $j$ gives

$$\Delta^{k'}_{j'i'} = \frac{\partial^2 x^k}{\partial x^{j'} \partial x^{i'}} \frac{\partial x^{k'}}{\partial x^k} + \frac{\partial x^i}{\partial x^{j'}} \frac{\partial x^j}{\partial x^{i'}} \frac{\partial x^{k'}}{\partial x^k} \Delta^k_{ji}. \qquad (18)$$

Subtracting (18) from (17) gives

$$\Delta^{k'}_{[i'j']} = \frac{\partial x^i}{\partial x^{j'}} \frac{\partial x^j}{\partial x^{i'}} \frac{\partial x^{k'}}{\partial x^k} \Delta^k_{[ij]}. \qquad (19)$$

It follows from the relationships (16) and (13) that the connection of absolute parallelism features the *Ricci torsion*

$$\Delta^k_{[ij]} = -\Omega^{..k}_{ij}, \qquad (20)$$

defined by the object of anholonomity.

# 2 Covariant differentiation in $A_4$ geometry. Ricci rotation coefficients

The definition of the covariant derivative with respect to the connection of the geometry of absolute parallelism ($A_4$ geometry) $\Delta^i_{jk}$ from a tensor of arbitrary valence $U^{i\ldots p}_{m\ldots n}$ has the form

$$\begin{array}{c} \overset{*}{\nabla}_k U^{i\ldots p}_{m\ldots n} = U^{i\ldots p}_{m\ldots n,k} + \Delta^i_{jk} U^{j\ldots p}_{m\ldots n} + \ldots + \Delta^p_{jk} U^{i\ldots j}_{m\ldots n} - \\ \Delta^j_{mk} U^{i\ldots p}_{j\ldots n} - \ldots - \Delta^j_{nk} U^{i\ldots p}_{m\ldots j}. \end{array} \qquad (21)$$

This definition enables some quite useful relationships in $A_4$ geometry to be proved.

**Proposition 5.1.** Parallel displacement of the tetrad $e^a{}_i$ relative to the connection $\Delta^i_{jk}$ equals zero identically.

**Proof.** From the definition (21) we have the following equalities:

$$\overset{*}{\nabla}_k \, e^i{}_a = a^i{}_{a,k} + \Delta^i_{jk} e^j{}_a, \qquad (22)$$

$$\overset{*}{\nabla}_k \, e^a{}_j = e^a{}_{j,k} - \Delta^i_{jk} e^a{}_i. \qquad (23)$$

Since the connection $\Delta^i_{jk}$ is defined as

$$\Delta^i_{jk} = e^i{}_a e^a{}_{j,k}, \qquad (24)$$

we have

$$e^i{}_a e^a{}_{j,k} - \Delta^i_{jk} = 0.$$

Multiplying this equality by $e^a{}_i$ and taking into consideration the orthogonality conditions (5.1), we get

$$\overset{*}{\nabla}_k \, e^a{}_j = e^a{}_{j,k} - \Delta^i_{jk} e^a{}_i = 0. \tag{25}$$

To prove that the relationship (22) is zero, we will take a derivative of the convolution $e^a{}_j e^i{}_a = \delta^i_j$

$$(\delta^i_j)_{,k} = (e^a_j e^i_a)_{,k} = e^i{}_a e^a{}_{j,k} + e^a{}_j e^i_{a,k} = 0.$$

Hence, by (24), we have

$$\Delta^i_{jk} = -e^a{}_j e^i{}_{a,k} \tag{26}$$

or

$$e^a{}_j e^i{}_{a,k} + \Delta^i_{jk} = 0.$$

Multiplying this relationship by $e^j{}_a$ and using the conditions $e^a{}_j e^i{}_a = \delta^i_j$, we have

$$\overset{*}{\nabla}_k \, e^i{}_a = e^i_{a,k} + \Delta^i_{jk} e^j{}_a = 0. \tag{27}$$

**Proposition 5.2.** Connection $\Delta^i_{jk}$ can be represented as the sum

$$\Delta^i_{jk} = \Gamma^i_{jk} + T^i_{jk}, \tag{28}$$

where $\Gamma^i_{jk}$ are the Christoffel symbols given by the relationship (6), and

$$T^i_{jk} = -\Omega^{..i}_{jk} + g^{im}(g_{js}\Omega^{..s}_{mk} + g_{ks}\Omega^{..s}_{mj}) \tag{29}$$

are *the Ricci rotation coefficients* [2].

**Proof.** Let us represent the connection (28) as the sum of parts symmetrical and skew-symmetrical in indices $j, k$

$$\Delta^i_{jk} = \Delta^i_{(jk)} + \Delta^i_{[jk]}, \tag{30}$$

where

$$\Delta^i_{(jk)} = \frac{1}{2}(\Delta^i_{jk} + \Delta^i_{kj}), \quad \Delta^i_{[jk]} = \frac{1}{2}(\Delta^i_{jk} - \Delta^i_{jk}).$$

We now add to and subtract from the right-hand side of (30) the same expression

$$\begin{gathered} \Delta^i_{jk} = \Delta^i_{(jk)} + \Delta^i_{[jk]} + g^{im}(g_{js}\Delta^s_{[km]} + g_{ks}\Delta^s_{[jm]}) - \\ g^{im}(g_{js}\Delta^s_{[km]} + g_{ks}\Delta^s_{[jm]}). \end{gathered} \tag{31}$$

We then group the terms on the right-hand side of (31) as follows:

$$\begin{aligned} \Delta^i_{jk} &= \Delta^i_{(jk)} - g^{im}(g_{js}\Delta^s_{[km]} + g_{ks}\Delta^s_{[jm]}) + \\ &+ \Delta^i_{[jk]} + g^{im}(g_{js}\Delta^s_{[km]} + g_{ks}\Delta^s_{[jm]}). \end{aligned} \tag{32}$$

Since

$$\Delta^i_{[jk]} = -\Omega^{..i}_{jk},$$

it follows from (32) and (29) that

$$\Delta^i_{jk} = \Delta^i_{(jk)} - g^{im}(g_{js}\Delta^s_{[km]} + g_{ks}\Delta^s_{[jm]}) + T^i_{jk}. \tag{33}$$

We now show that

$$\Gamma^i_{jk} = \Delta^i_{(jk)} - g^{im}(g_{js}\Delta^s_{[km]} + g_{ks}\Delta^s_{[jm]}). \tag{34}$$

Actually, we have the relationships

$$\Delta^i_{(jk)} = e^i{}_a e^a_{(j,k)} = \frac{1}{2} e^i{}_a (e^a{}_{j,k} + e^a{}_{k,j}),$$

$$\Delta^i_{[jk]} = e^i{}_a e^a_{[j,k]} = \frac{1}{2} e^i{}_a (e^a{}_{j,k} - e^a{}_{k,j}),$$

$$g_{js} = \eta_{ab} e^a{}_j e^b{}_s, \tag{35}$$

therefore (34) become

$$\Gamma^i_{jk} = e^i{}_a e^a_{(j,k)} + g^{im}(\eta_{ab} e^a{}_j e^b_{[m,k]} + \eta_{ab} e^a_k e^b_{[m,j]}) =$$
$$= \frac{1}{2}\eta^{cd}\eta_{ab} e^i_c e^m_d (e^b_m e^a{}_{j,k} + e^b_m e^c{}_{k,j}) +$$
$$+\frac{1}{2} g^{im}\left(\eta_{ab}(e^a{}_j e^b{}_{m,k} - e^a{}_j e^b{}_{k,m}) + \eta_{ab}(e^a_k e^b{}_{m,j} - e^a_k e^b{}_{j,m})\right).$$

Regrouping the terms here gives

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}\left((\eta_{ab} e^a{}_j e^b{}_m)_{,k} + (\eta_{ab} e^a_k e^b_m)_{,j} - (\eta_{ab} e^a{}_j e^b{}_k)_{,m}\right).$$

Hence, by (35), we obtain

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m}), \tag{36}$$

or

$$\frac{1}{2} g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m}) =$$
$$= \Delta^i_{(jk)} - g^{im}(g_{js}\Delta^s_{[km]} + g_{ks}\Delta^s_{[jm]}) = \Gamma^i_{jk}. \tag{37}$$

Substituting (37) into (33), we get the relationship (28).

**Proposition 5.3.** The Ricci rotation coefficients $T^i_{jk}$ can be represented in the form

$$T^i_{jk} = e^i{}_a \nabla_k e^a{}_j, \tag{38}$$

$$T^i_{jk} = -e^a{}_j \nabla_k e^i{}_a, \tag{39}$$

where $\nabla_k$ stands for a covariant derivative with respect to the Christoffel $\Gamma^i_{jk}$ symbols.
**Proof.** We will represent in the relationships (25) and (27) the connection $\Delta^i_{jk}$ as the sum (28)

$$\overset{*}{\nabla}_k\, e^a{}_j = e^a{}_{j,k} - \Gamma^i_{jk} e^a{}_i - T^i_{jk} e^a{}_i = 0, \tag{40}$$

$$\overset{*}{\nabla}_k\, e^i{}_a = e^i_{a,k} + \Gamma^i_{jk} e^j{}_a + T^i_{jk} e^j{}_a = 0 \tag{41}$$

Since, by definition [1], we can write

$$\nabla_k e^a{}_j = e^a{}_{j,k} - \Gamma^i_{jk} e^a{}_i,$$

$$\nabla_k e^i{}_a = e^i_{a,k} + \Gamma^i_{jk} e^j{}_a,$$

then (40) and (41) can be written as

$$\nabla_k e^a{}_j - T^i_{jk} e^a{}_i = 0, \tag{42}$$

$$\nabla_k e^i{}_a + T^i_{jk} e^j{}_a = 0. \tag{43}$$

Multiplying (42) by $e^i{}_a$ and (43) by $e^a{}_j$, respectively, we will obtain (using the orthogonality conditions (1)), by (42), (43), the relationships (38) and (39).

We will now calculate the covariant derivative $\overset{*}{\nabla}_k$ with respect to the metric tensor $g^{jm}$, knowing that $g^{jm} = \eta^{ab} e^j{}_a e^m_b$

$$\overset{*}{\nabla}_k\, g^{jm} = \overset{*}{\nabla}_k\, \eta^{ab} e^j{}_a e^m_b = \overset{*}{\nabla}_k\, e^j{}_a e^{ma} =$$
$$= e^{ma}\, \overset{*}{\nabla}_k\, e^j{}_a + e^j{}_a\, \overset{*}{\nabla}_k\, e^{ma}.$$

From the relationships (25) and (27), we have

$$\overset{*}{\nabla}\, g^{jm} = 0. \tag{44}$$

On the other hand, applying the formula (21) to the relationship (44), we find that

$$\overset{*}{\nabla}_k\, g^{jm} = g^{jm}_{,k} + \Delta^j_{pk} g^{pm} + \Delta^m_{pk} g^{jp} = 0. \tag{45}$$

Substituting the connection $\Delta^i_{jk}$ as the sum (28), we will write the relationship (45) in the form

$$\overset{*}{\nabla}_k\, g^{jm} = \nabla_k g^{jm} + T^j_{pk} g^{pm} + T^m_{pk} g^{jp} = 0. \tag{46}$$

From the equality

$$\nabla_k g^{jm} = g^{jm}_{,k} + \Gamma^j_{pk} g^{pm} + \Gamma^m_{pk} g^{jp} = 0, \tag{47}$$

we have, by (46),

$$T^j_{pk} g^{pm} + T^m_{pk} g^{jp} = T^{jm}_k + T^{mj}_k = 0.$$

This equality establishes the following symmetry properties for the Ricci rotation coefficients:

$$T_{jmk} = -T_{mjk}. \tag{48}$$

Therefore, in the $A_4$ geometry the Ricci rotation coefficients have 24 independent components.

## 3 Curvature tensor of $\mathbf{A}_4$ space

The curvature tensor of the space of absolute parallelism $S^i{}_{jkm}$ is defined in terms of the connection $\Delta^i_{jk}$ following a conventional rule [18]

$$S^i{}_{jkm} = 2\Delta^i_{j[m,k]} + 2\Delta^i_{s[k}\Delta^s_{|j|m]} = 0, \tag{49}$$

where the parentheses [ ] signify alternation in appropriate indices, whereas the index within the vertical lines | | is not subject to alternation.

**Proposition 5.4.** The Riemann-Christoffel tensor of a space with the connection (26) equals zero identically.
**Proof.** From the relationship (26) we have

$$e^a{}_{j,k} = \Delta^i_{jk} e^a{}_i. \tag{50}$$

Differentiating the relationship (50) with respect to $m$ gives

$$\begin{aligned} e^a{}_{j,k,m} = (\Delta^i_{jk} e^a{}_i)_{,m} = \Delta^i_{jk,m} e^a{}_i + e^a_{i,m}\Delta^i_{jk} = \\ = (\Delta^i_{jk,m} + e^i{}_a e^a_{s,m}\Delta^s_{jk}) e^a{}_i = (\Delta^i_{jk,m} + \Delta^i_{sm}\Delta^s_{jk}) e^a{}_i. \end{aligned}$$

Alternating this relationship in indices $k$ and $m$ we get

$$-2e^a_{j,[k,m]} = 2(\Delta^i_{j[m,k]} + 2\Delta^i_{s[k}\Delta^s_{|j|m]}) = S^i{}_{jkm} e^a{}_i. \tag{51}$$

Since the operation of differentiating with respect to indices $k$ and $m$ is symmetrical, we have

$$e^a_{j,[k,m]} = 0,$$

From this equality, considering that $e^a{}_i$ in (51) is arbitrary, we will get

$$S^i{}_{jkm} = 0. \tag{52}$$

**Proposition 5.5.** Tensor $S^i{}_{jkm}$ can be represented as the sum

$$S^i{}_{jkm} = R^i{}_{jkm} + 2\nabla_{[k} T^i_{|j|m]} + 2T^i_{c[k} T^c_{|j|m]} = 0, \tag{53}$$

where

$$R^i{}_{jkm} = 2\Gamma^i_{j[m,k]} + 2\Gamma^i_{s[k}\Gamma^s_{|j|m]} \tag{54}$$

is the tensor of the Riemannian space $A_4$.
**Proof.** Substituting the sum $\Delta^i_{jk} = \Gamma^i_{jk} + T^i_{jk}$ into (49) gives

$$\begin{aligned} S^i{}_{jkm} = 2\Gamma^i_{j[m,k]} + 2\Gamma^i_{s[k}\Gamma^s_{|j|m]} + 2T^i_{j[m,k]} + 2T^i_{s[k}T^s_{|j|m]} + \\ 2T^i_{s[k}\Gamma^s_{|j|m]} + 2\Gamma^i_{s[k}T^s_{|j|m]} = 0. \end{aligned} \tag{55}$$

Using (54), we will write (55) as follows:

$$\begin{aligned} S^i{}_{jkm} = R^i{}_{jkm} + 2T^i_{j[m,k]} + 2T^i_{s[k}T^s_{|j|m]} + \\ +2\Gamma^s_{j[k}T^i_{|s|m]} + 2\Gamma^i_{s[k}T^s_{|j|m]} = 0. \end{aligned} \tag{56}$$

If now we add to the right-hand side of this relationship the expression

$$-2\Gamma^{s}_{[km]}T^{i}_{sj} = 0,$$

and take into consideration that [1]

$$\nabla_k U^{i...p}_{m...n} = U^{i...p}_{m...n,k} + \Gamma^{i}_{jk}U^{j...p}_{m...n} + \ldots + \Gamma^{p}_{jk}U^{i...j}_{m...n} - \Gamma^{j}_{mk}U^{i...p}_{j...n} - \ldots - \Gamma^{j}_{nk}U^{i...p}_{m...j}, \tag{57}$$

we will obtain from (56) the equality (53).

Let us now rewrite the relationship (53) as

$$R^{i}{}_{jkm} = -2T^{i}_{j[m,k]} - 2T^{i}_{s[k}T^{s}_{|j|m]}. \tag{58}$$

Substituting here (38) and (39)

$$T^{i}_{jk} = e^{i}{}_{a}\nabla_k e^{a}{}_{j}, \quad T^{i}_{jk} = -e^{a}{}_{j}\nabla_k e^{i}{}_{a},$$

we obtain

$$-2T^{i}_{j[m,k]} = -2e^{i}{}_{a}\nabla_{[k}\nabla_{m]}e^{a}{}_{j} - 2\nabla_{[k}e^{i}_{|a|}\nabla_{m]}e^{a}{}_{j},$$

$$-2T^{i}_{s[k}T^{s}_{|j|m]} = 2e^{a}{}_{s}\nabla_{[k}e^{i}_{|a}e^{s}{}_{a|}\nabla_{m]}e^{a}{}_{j} = 2\nabla_{[k}e^{i}_{|a|}\nabla_{m]}e^{a}{}_{j}.$$

Therefore, it follows from the relationships (58) that

$$R^{i}{}_{jkm} = -2e^{i}{}_{a}\nabla_{[k}\nabla_{m]}e^{a}{}_{j} = 2e^{i}{}_{a}\nabla_{[m}\nabla_{k]}e^{a}{}_{j}. \tag{59}$$

**Proposition 5.6.** The torsion field $\Omega^{..i}_{jk}$ of the $A_4$ space satisfies the equations

$$\overset{*}{\nabla}_{[k}\ \Omega^{..i}_{jm]} + 2\Omega^{..s}_{[kj}\Omega^{..i}_{m]s} = 0. \tag{60}$$

**Proof.** Alternating the expression (49) in indices $j,k,m$ and using the relationship $\Delta^{i}_{[jk]} = -\Omega^{..i}_{jk}$, we get

$$S^{i}_{[jkm]} = 2\Omega^{..i}_{[jm,k]} + 2\Delta^{i}_{s[k}\Omega^{..s}_{jm]} = 0. \tag{61}$$

If then we add and subtract here the quantity

$$2\Delta^{s}_{[kj}\Omega^{..i}_{|s|m]} + 2\Delta^{s}_{[km}\Omega^{..i}_{j]s},$$

we will have

$$2\Omega^{..i}_{[jm,k]} + 2\Delta^{i}_{s[k}\Omega^{..s}_{jm]} - 2\Delta^{s}_{[kj}\Omega^{..i}_{|s|m]} - 2\Delta^{s}_{[km}\Omega^{..i}_{j]s} + 2\Delta^{s}_{[kj}\Omega^{..i}_{|s|m]} + 2\Delta^{s}_{[km}\Omega^{..i}_{j]s} = 0.$$

Using the formula (21), we can rewrite this relationship as follows:

$$2\overset{*}{\nabla}_{[k}\ \Omega^{..i}_{jm]} - 2\Omega^{..s}_{[kj}\Omega^{..i}_{|s|m]} - 2\Omega^{..s}_{[km}\Omega^{..i}_{j]s} = 2\overset{*}{\nabla}_{[k}\ \Omega^{..i}_{jm]} + 4\Omega^{..s}_{[kj}\Omega^{..i}_{m]s} = 0, \tag{62}$$

whence we have (60).

**Proposition 5.7.** The Riemann tensor $R^i{}_{jkm}$ of the $A_4$ space satisfies the equality

$$R^i_{[jkm]} = 0. \tag{63}$$

**Proof.** Alternating the relationship (54) in indices $j, k, m$ and using the equality

$$T^i_{[jk]} = -\Omega^{\cdot\cdot i}_{jk},$$

we have

$$R^i_{[jkm]} = 2\nabla_{[k}\Omega^{\cdot\cdot i}_{jm]} + 2T^i_{s[k}\Omega^{\cdot\cdot s}_{jm]}.$$

If in the right-hand side of the equality we add and subtract the quantity

$$2T^s_{[kj}\Omega^{\cdot\cdot i}_{|s|m]} + 2T^s_{[km}\Omega^{\cdot\cdot i}_{j]s},$$

we obtain

$$R^i_{[jkm]} = 2\nabla_{[k}\Omega^{\cdot\cdot i}_{jm]} + 2T^i_{s[k}\Omega^{\cdot\cdot s}_{jm]} - 2T^s_{[kj}\Omega^{\cdot\cdot i}_{|s|m]} - 2T^s_{[km}\Omega^{\cdot\cdot i}_{j]s} +$$
$$+2T^s_{[kj}\Omega^{\cdot\cdot i}_{|s|m]} + 2T^s_{[km}\Omega^{\cdot\cdot i}_{j]s} = 2\overset{*}{\nabla}_{[k}\Omega^{\cdot\cdot i}_{jm]} - 2\Omega^{\cdot\cdot s}_{[kj}\Omega^{\cdot\cdot i}_{|s|m]} -$$
$$-2\Omega^{\cdot\cdot s}_{[km}\Omega^{\cdot\cdot i}_{j]s} = 2\overset{*}{\nabla}_{[k}\Omega^{\cdot\cdot i}_{jm]} + 4\Omega^{\cdot\cdot s}_{[kj}\Omega^{\cdot\cdot i}_{m]s} = 0,$$

which proves the validity of the relationship (63).

# 4 Formalism of external forms and the matrix treatment of Cartan's structural equations of the absolute parallelism geometry

Consider the differentials

$$dx^i = e^a e^i{}_a, \tag{64}$$

$$de^i{}_b = \Delta^a{}_b e^i{}_a, \tag{65}$$

where

$$e^a = e^a_i dx^i, \tag{66}$$

$$\Delta^a{}_b = e^a{}_i de^i{}_b = \Delta^a{}_{bk} dx^k \tag{67}$$

are differential 1-forms of tetrad $e^a_i$ and connection of absolute parallelism $\Delta^a{}_{bk}$. Differentiating the relationships (64), (65) externally [3], we have, respectively,

$$d(dx^i) = (de^a - e^c \wedge \Delta^a{}_c)e^i{}_a = -S^a e^i{}_a, \tag{68}$$

$$d(de^i{}_a) = (d\Delta^b{}_a - \Delta^c{}_a \wedge \Delta^b{}_c)e^i{}_b = -S^b{}_a e^i{}_b. \tag{69}$$

Here $S^a$ denotes the 2-form of *the Cartan torsion* [3], and $S^b{}_a$ – the 2-form of the curvature tensor. The sign $\wedge$ signifies external product, e.g,

$$e^a \wedge e^b = e^a e^b - e^b e^a. \tag{70}$$

By definition, a space has a geometry of absolute parallelism, if the 2-form of Cartanian torsion $S^a$ and the 2-form of the Riemann-Christoffel curvature $S^b{}_a$ of this space vanishe

$$S^a = 0, \tag{71}$$

$$S^b{}_a = 0. \tag{72}$$

At the same time, these equalities are the integration conditions for the differentials (64) and (65).

Equations

$$de^a - e^c \wedge \Delta^a{}_c = -S^a, \tag{73}$$

$$d\Delta^b{}_a - \Delta^c{}_a \wedge \Delta^b{}_c = -S^b{}_a, \tag{74}$$

which follow from (68) and (69), are Cartan's structural equations for an appropriate geometry. For the geometry of absolute parallelism hold the conditions (71) and (72), therefore Cartan's structural equations for $A_4$ geometry have the form

$$de^a - e^c \wedge \Delta^a{}_c = 0, \tag{75}$$

$$d\Delta^b{}_a - \Delta^c{}_a \wedge \Delta^b{}_c = 0. \tag{76}$$

Considering (28), we will represent 1-form $\Delta^a{}_b$ as the sum

$$\Delta^a{}_b = \Gamma^a{}_b + T^a_b. \tag{77}$$

Substituting this relationship into (75) and noting that

$$e^c \wedge \Delta^a{}_c = e^c \wedge T^a{}_c,$$

we get the first of Cartan's structural equations for $A_4$ space.

$$de^a - e^c \wedge T^a{}_c = 0. \tag{A}$$

Substituting (77) into (76) gives the second of Cartan's equations for $A_4$ space.

$$R^a_b + dT^a{}_b - T^c{}_b \wedge T^a{}_c = 0, \tag{B}$$

where $R^a{}_b$ is the 2-form of the Riemann tensor

$$R^a{}_b = d\Gamma^a{}_b - \Gamma^c{}_b \wedge \Gamma^a{}_c. \tag{78}$$

By definition [3], we always have the relationships

$$dd(dx^i) = 0, \tag{79}$$

$$dd(de^i{}_a) = 0. \tag{80}$$

In the geometry of absolute parallelism these equalities become

$$d(de^a - e^c \wedge T^a{}_c) = R^a{}_{cfd} e^c \wedge e^f \wedge e^d = 0, \tag{81}$$

$$d(R^a{}_b + dT^a{}_b - T^c{}_b \wedge T^a{}_c) = dR^a{}_b + R^f{}_b \wedge T^a{}_f - T^f{}_b \wedge R^a{}_f = 0. \tag{82}$$

Here

$$R^a{}_{cfd} = -2T^a{}_{c[d,f]} - 2T^a{}_{b[f}T^b{}_{|c|d]}.$$

Equalities (81) and (82) represent the first and second of Bianchi's identities, respectively, for $A_4$ space. Dropping the indices, we can write Cartan's structural equations and Bianchi's identities for the $A_4$ geometry as

$$\boxed{\begin{array}{cc} de - e\wedge T = 0, & (A)\\ R + dT - T\wedge T = 0, & (B)\\ R\wedge e\wedge e\wedge e = 0, & (C)\\ dR + R\wedge T - T\wedge R = 0. & (D)\end{array}}$$

**Proposition 5.8.** The matrix treatment of the first of Cartan's structural equations $(A)$ of the $A_4$ geometry has the form

$$\nabla_{[k}e^a{}_{m]} - e^b{}_{[k}T^a{}_{|b|m]} = 0. \tag{83}$$

**Proof.** Let us write equations $(A)$ as

$$de^a - e^c\wedge T^a_{\ c} = 0. \tag{84}$$

Further, by (66), we have

$$de^a = d(e^a{}_m dx^m) = \nabla_k e^a{}_m dx^k\wedge dx^m = \frac{1}{2}(\nabla_k e^a{}_m - \nabla_m e^a{}_k)dx^k\wedge dx^m$$

and, also,

$$e^b\wedge T^a_{\ b} = e^b{}_k T^a_{\ bm}dx^k\wedge dx^m = \frac{1}{2}(e^b{}_k T^a_{\ bm} - e^b{}_m T^a_{\ bk})dx^k\wedge dx^m.$$

Substituting these relationships into equations (84) we will derive the matrix equations in the form

$$\nabla_{[k}e^a{}_{m]} - e^b{}_{[k}T^a{}_{|b|m]} = 0, \tag{A}$$

where the matrixes $e^a{}_m$ and $T^a_{\ bm}$ in world indices $i, j, m, \ldots$ are transformed as vectors

$$e^a{}_{m'} = \frac{\partial x^m}{\partial x^{m'}}e^a{}_m, \tag{85}$$

$$T^a_{\ bm'} = \frac{\partial x^m}{\partial x^{m'}}T^a_{\ bm}, \tag{86}$$

and in the matrix indices $a, b, c, \ldots$ they are transformed as follows:

$$e^{a'}{}_m = \Lambda_a{}^{a'}e^a{}_m, \tag{87}$$

$$T^{a'}_{\ b'k} = \Lambda_a{}^{a'}T^a_{\ bk}\Lambda^b{}_{b'} + \Lambda^{a'}{}_a\Lambda^a{}_{b',k}. \tag{88}$$

In relationships (87) and (88) the matrices $\partial x^{m'}/\partial x^m$ form a translation group $T_4$ that is defined on a manifold of world coordinates $x^i$. On the other hand, the matrices $\Lambda^{a'}{}_a$ form a group of four-dimensional rotations $O(3.1)$

$$\Lambda^{a'}_a \in O(3.1),$$

defined on the manifold of "angular coordinates" $e^a{}_i$. Actually, the tetrad $e^a{}_i$ is a mathematical image of an arbitrarily accelerated four-dimensional reference frame. Such a frame has ten degrees of freedom: four translational ones connected with the motion of its origin, and six angular ones describing variations of its orientation. The six independent components of the tetrad $e^a{}_i$ represent six direction cosines of six independent angles defining the orientation of the tetrad in space.

**Proposition 5.9.** The matrix rendering of the second of Cartan's structuring equations $(B)$ of the $A_4$ geometry has the form

$$R^a{}_{bkm} + 2\nabla_{[k}T^a{}_{|b|m]} + 2T^a{}_{c[k}T^c{}_{|b|m]} = 0. \tag{89}$$

**Proof.** We will expand the 2-form $R^a{}_d$ as

$$R^a{}_b = \frac{1}{2}R^a{}_{bcd}e^c \wedge e^d = \frac{1}{2}R^a{}_{bkm}dx^k \wedge dx^m. \tag{90}$$

Further, we have

$$dT^a_b = d(T^a_{bm}dx^m) = \nabla_k T^a_{bm}dx^k \wedge dx^m = \frac{1}{2}(\nabla_k T^a_{bm} - \nabla_m T^a_{bk})dx^k \wedge dx^m, \tag{91}$$

and also

$$T^a_c \wedge T^c_b = T^a_{ck}T^c_{bm}dx^k \wedge dx^m = = \frac{1}{2}(T^a_{ck}T^c_{bm} - T^c_{bm}T^a_{ck})dx^k \wedge dx^m. \tag{92}$$

Let us substitute the relationships (92)–(94) into

$$R^a_b + dT^a_b - T^c_b \wedge T^a_c = 0.$$

Simple transformations yield

$$\frac{1}{2}(R^a{}_{bkm} + \nabla_k T^a_{bm} - \nabla_m T^a_{bk} + T^a_{ck}T^c_{bm} - T^c_{bm}T^a_{ck})dx^k \wedge dx^m = 0.$$

Since here the factor $dx^k \wedge dx^m$ is arbitrary, we have

$$R^a{}_{bkm} + \nabla_k T^a_{bm} - \nabla_m T^a_{bk} + T^a_{ck}T^c_{bm} - T^c_{bm}T^a_{ck} = 0,$$

which is equivalent to the equations (89).

**Proposition 5.10.** The matrix form of the Bianchi identity $(D)$ of $A_4$ geometry is

$$\nabla_{[n}R^a{}_{|b|km]} + R^c{}_{b[km}T^a{}_{|c|n]} - T^c{}_{b[n}R^a{}_{|c|km]} = 0. \tag{93}$$

**Proof.** The external differential $dR^a{}_b$ in the identities $(D)$ has the 2-form

$$dR^a_b = \frac{1}{2}\nabla_n R^a{}_{bkm}dx^n \wedge dx^k \wedge dx^m = = \frac{1}{6}(\nabla_n R^a{}_{bkm} + \nabla_m R^a{}_{bkn} + \nabla_k R^a{}_{bmn})dx^n \wedge dx^k \wedge dx^m. \tag{94}$$

In addition, we have

$$R^f{}_b \wedge T^a{}_f = \frac{1}{2} R^f{}_{bkm} T^a{}_{fn} dx^k \wedge dx^m \wedge dx^n =$$
$$= \frac{1}{6}(R^f{}_{bkm} T^a{}_{fn} + R^f{}_{bnk} T^a{}_{fm} + R^f{}_{bmn} T^a{}_{fk}) dx^k \wedge dx^m \wedge dx^n, \quad (95)$$

$$T^f{}_b \wedge R^a{}_f = \frac{1}{2} T^f{}_{bn} R^a{}_{fkm} dx^n \wedge dx^k \wedge dx^m =$$
$$= \frac{1}{6}(T^f{}_{bn} R^a{}_{fkm} + T^f{}_{bm} R^a{}_{fnk} + T^f{}_{bk} R^a{}_{fmn}) dx^n \wedge dx^k \wedge dx^m. \quad (96)$$

Substituting relationships (94)–(96) into the identity

$$dR^a{}_b + R^f{}_b \wedge T^a{}_f - T^f{}_b \wedge R^a{}_f = 0$$

and considering that $dx^n \wedge d^k \wedge dx^m$ is arbitrary, we get

$$\nabla_n R^a{}_{bkm} + \nabla_m R^a{}_{bkn} + \nabla_k R^a{}_{bmn} + R^f{}_{bkm} T^a{}_{fn} + R^f{}_{bnk} T^a{}_{fm} +$$
$$+ R^f{}_{bmn} T^a{}_{fk} - T^f{}_{bn} R^a{}_{fkm} - T^f{}_{bm} R^a{}_{fnk} - T^f{}_{bk} R^a{}_{fmn} = 0,$$

which is equivalent to the identity (93).

The first of Bianchi's identities ($C$) of $A_4$ geometry in indices of the group $O(3.1)$ is written as

$$R^a{}_{[bcd]} = 0, \quad (97)$$

or, which is the same, as

$$\overset{*}{\nabla}_{[b}\, \Omega^{..a}_{cd]} + 2\Omega^{..f}_{[bc} \Omega^{..a}_{d]f} = 0. \quad (98)$$

## 5 $\mathbf{A_4}$ geometry as a group manifold. Killing-Cartan metric

The matrix representation of Cartan's structural equations of the geometry of absolute parallelism indicates that, in fact, this space behaves as a manifold, on which the translations group $T_4$ and the rotations group $O(3.1)$ are specified. We will consider $A_4$ geometry as a group 10-dimensional manifold formed by four translational coordinates $x_i$ $(i = 0,1,2,3)$ and six (by the relationship $e^a{}_i e^j{}_a = \delta_i{}^j$) angular coordinates $e^a{}_i$ $(a = 0,1,2,3)$. Suppose that on this manifold a group of four-dimensional translations $T_4$ and a rotations group $O(3.1)$ are defined. We then introduce the Hayashi invariant derivative [4]

$$\nabla_b = e^k{}_b \partial_k, \quad (99)$$

whose components are generators of the translations group $T_4$ that is specified on the manifold of translational coordinates $x_i$. If then we represent as a sum

$$e^k{}_b = \delta^k{}_b + a^k{}_b, \quad (100)$$

$$i, j, k \ldots = 0,1,2,3, \quad a, b, c, \ldots = 0,1,2,3,$$

then the field $a^k{}_b$ can be viewed as the potential of the gauge field of the translations group $T_4$ [4]. In the case where $a^k{}_b = 0$, the generators (99) coincide with the generators of the translations group of the pseudo-Euclidean space $E_4$ .

We know already that in the coordinate index $k$ the nonholonomic tetrad $e^k{}_a$ transforms as the vector

$$e^{k'}{}_a = \frac{\partial x^{k'}}{\partial x^k} e^k{}_a,$$

whence, by (100), we have the law of transformation for the field $a^k{}_a$ relative to the translationss

$$a^{k'}{}_b = \frac{\partial x^{k'}}{\partial x^n} a^n{}_b + \frac{\partial x^{k'}}{\partial x^n} \delta^n{}_b - \delta^{k'}{}_b. \tag{101}$$

We define the tetrad $e^i{}_a$ as

$$e^i{}_a = \nabla_a x^i \tag{102}$$

and write the commutational relationships for the generators (99) as

$$\nabla_{[a} \nabla_{b]} = -\Omega_{ab}^{..c} \nabla_c, \tag{103}$$

where $-\Omega_{ab}^{..c}$ are the structural functions for the translations group of the space$A_4$. If then we apply the operator (103) to the manifold $x^i$, we will arrive at the structural equations of the group $T_4$ of the space $A_4$ as

$$\nabla_{[a} \nabla_{b]} x^i = -\Omega_{ab}^{..c} \nabla_c x^i \tag{104}$$

or

$$\nabla_{[a} e^i{}_{b]} = -\Omega_{ab}^{..c} e^i{}_c. \tag{105}$$

In this relationship the structural functions $-\Omega_{ab}^{..c}$ are defined as

$$-\Omega_{ab}^{..c} = e^c{}_i \nabla_{[a} e^i{}_{b]}. \tag{106}$$

It is seen from this equality that when the potentials of the gauge field of translations group $a_b^k$ in the relationship (100) vanish, so do the structural functions (106). Therefore, we will refer to the field $\Omega_{ab}^{..c}$ as the gauge field of the translations group.

Considering that $T^c{}_{[ab]} = -\Omega_{ab}^{..c}$, we will rewrite the structural equations (106) as

$$\nabla_{[k} e^a{}_{m]} - e^b{}_{[k} T^a{}_{|b|m]} = 0. \tag{107}$$

It is easily seen that the equations (107) can be derived by alternating the equations (42). What is more, they coincide with the structural Cartan equations (A) of the geometry of absolute parallelism.

The structural equations of group $T_4$, written as (106), can be regarded as a definition for the torsion of space $A_4$. So the torsion of space $A_4$ coincides with the structural function of the translations group of this space, such that the structural functions obey the generalized Jacobi identity

$$\overset{*}{\nabla}_{[b}\, \Omega_{cd]}^{..a} + 2\Omega_{[bc}^{..f} \Omega_{d]f}^{..a} = 0, \tag{108}$$

where $\overset{*}{\nabla}_b$ is the covariant derivative with respect to the connection of absolute parallelism $\Delta^a_{bc}$. Comparing the identity (108) with the Bianchi identity (98) of the geometry $A_4$, we see that we deal with the same identity. The Jacobi identity (5.108), which is obeyed by the structural functions of the translations group of geometry $A_4$, coincides with the first Bianchi identity of the geometry of absolute parallelism .

The vectors

$$e^i{}_a = \nabla_a x^i, \tag{109}$$

that form the vector stratification [3] of the $A_4$ geometry, point along the tangents to each point of the manifold $x^i$ of the pseudo-Euclidean plane with the metric tensor

$$\eta_{ab} = \eta^{ab} = \mathrm{diag}(1,\,-1,\,-1,\,-1). \tag{110}$$

Therefore, the ten-dimensional manifold (four translational coordinates $x^i$ and six "rotational" coordinates $e^i{}_a$) of the geometry of absolute parallelism can be regarded as the stratification with the coordinates of the base $x^i$ and the (anholonomic) "coordinates" of the fibre $e^i{}_c$. If on the base $x^i$ we have the translations group $T_4$, then in the fibre $e^i{}_c$ we have the rotation group $O(3.1)$. It follows from (109) that the infinitesimal translations in the base $x^i$ in the direction $a$ are given by the vector

$$ds^a = e^a{}_i dx^i. \tag{111}$$

If from (111) and the covariant vector $ds_a = e^i{}_a dx_i$ we form the invariant convolution $ds^2$, we will obtain the Riemannian metric of $A_4$ space

$$ds^2 = g_{ik} dx^i dx^k \tag{112}$$

with the metric tensor

$$g_{ik} = \eta_{ab} e^a{}_i e^b{}_k.$$

Therefore, the Riemannian metric (112) can be viewed as the metric defined on the translations group $T_4$.

Since in the fibre we have the "angular coordinates" $e^i{}_a$ that form a manifold in which group $O(3.1)$ is defined, then it would be natural to define the structural equations for this group, as well as the metric specified on the group $O(3.1)$.

Let us rewrite the relationships (38) and (39) in matrix form

$$T^a_{\ bk} = e^a{}_i T^i{}_{jk} e^j{}_b = \nabla_k e^a{}_j e^j{}_b, \tag{113}$$

$$T^a_{\ bk} = e^a{}_i T^i{}_{jk} e^j{}_b = -e^a{}_i \nabla_k e^i{}_b. \tag{114}$$

These relationships enable the dependence between the infinitesimal rotation $d\chi_{ab} = -d\chi_{ba}$ of the vector $e^a{}_i$ at infinitesimal translations $ds_a$ to be established. In fact, by (113) and (114), we have

$$d\chi^a{}_b = T^a_{\ bk} dx^k = De^a{}_j e^j{}_b, \tag{115}$$

$$d\chi^a{}_b = T^a_{\ bk} dx^k = -e^a{}_i De^i{}_b. \tag{116}$$

where $D$ is the absolute differential [1] with respect to the Christoffel symbols $\Gamma^i_{jk}$. Using (115), we can form the invariant quadratic form $d\tau^2 = d\chi^a{}_b d\chi^b{}_a$ to arrive at the Killing-Cartan metric

$$d\tau^2 = d\chi^a{}_b d\chi^b{}_a = T^a_{\ bk} T^b_{\ an} dx^k dx^n = -De^a{}_i De^i{}_a \tag{117}$$

with the metric tensor

$$H_{kn} = T^{a}_{\ bk} T^{b}_{\ an}. \tag{118}$$

Unlike metric (112), the metric (117) is specified on the rotations group $O(3.1)$ that acts on the manifold of the "rotational coordinates" $e^{a}{}_{i}$.

Let us now introduce the covariant derivative

$$\overset{*}{\nabla}_m = \nabla_m + T_m, \tag{119}$$

where $T_m$ is the matrix $T^{a}_{bm}$ with discarded matrix indices. We will regard the components of the derivative as generators of the rotations group $O(3.1)$. Applying this operator to the tetrad $e^i$ that forms the manifold of "angular coordinates" of the $A_4$ geometry, we will arrive at

$$\overset{*}{\nabla}_m\, e^i = \nabla_m e^i + T_m e^i = 0, \tag{120}$$

hence

$$T_m = -e_i \nabla_m e^i. \tag{121}$$

It is interesting to note that, just as in (109) we have defined six "angular coordinates" $e^{i}{}_{a}$ through the four translational coordinates $x^i$, so in (5.121) we can define 24 "super-coordinates" $T^{a}_{bm}$ through the six coordinates $e^{i}{}_{a}$.

It follows from (120) that

$$\nabla_m e^i = -T_m e^i. \tag{122}$$

Recall that in the relationships (120)-(122) we have defined through $\nabla_m$ the covariant derivative with respect to $\Gamma^{i}_{jk}$. We will now take the covariant derivative $\nabla_k$ of the relationships (122)

$$\nabla_k \nabla_m e^i = -\nabla_k (T_m e^i) = -(\nabla_k T_m e^i + T_m \nabla_k e^i) =$$
$$= -(\nabla_k T_m e^i + T_m e^i e_i \nabla_k e^i).$$

Using (121), we will rewrite this expression as follows

$$\nabla_k \nabla_m e^i = -(\nabla_k T_m - T_m T_k) e^i.$$

Alternating this expression in the indices $k$ and $m$ gives

$$\nabla_{[k} \nabla_{m]} e^i = \frac{1}{2} R_{km} e^i, \tag{123}$$

where

$$R_{km} = 2\nabla_{[m} T_{k]} + [T_m,\, T_k]. \tag{124}$$

Introducing in equations (124) the matrix indices (the fibre indices), we will obtain the structural equation of the group $O(3.1)$

$$R^{a}{}_{bkm} = 2\nabla_{[m} T^{a}{}_{|b|k]} + 2T^{a}{}_{c[m} T^{c}{}_{|b|k]}. \tag{B}$$

It is easily seen that the structural equations of the rotations group $(B)$ coincide with the second of Cartan's structural equations (124) of the geometry $A_4$.

In this case the quantities $T^a{}_{bk}$ and $R^a{}_{bkm}$ vary in the rotations group $O(3.1)$ following the law

$$T^{a'}_{b'k} = \Lambda_a{}^{a'} T^a_{bk} \Lambda^b{}_{b'} + \Lambda_a{}^{a'} \Lambda^a{}_{b',k}, \tag{125}$$

and appear as the potentials of the gauge field $R^a{}_{bkm}$ of the rotations group $O(3.1)$. In the process, the gauge field of the group $O(3.1)$ obeys the formula

$$R^{a'}_{b'km} = \Lambda_a{}^{a'} R^a{}_{bkm} \Lambda^b{}_{b'}. \tag{126}$$

Note that the structural functions of the rotations group of $A_4$ geometry are the components of the curvature tensor $R^a{}_{bkm}$. It can be shown that the structural functions $R^a{}_{bkm}$ of the rotations group $O(3.1)$ satisfy the Jacobi identity

$$\nabla_{[n} R^a{}_{|b|km]} + R^c{}_{b[km} T^a{}_{|c|n]} - T^c{}_{b[n} R^a{}_{|c|km]} = 0, \tag{D}$$

which, at it was shown in the previous section, are at the same time the second Bianchi identities of the $A_4$ space.

Let us introduce the dual Riemann tensor

$$\overset{*}{R}_{ijkm} = \frac{1}{2} \varepsilon^{sp}{}_{km} R_{ijsp}, \tag{127}$$

where $\varepsilon^{sp}{}_{km}$ is the completely skew-symmetrical Levi-Chivita tensor. Then the equations (D) can be written as

$$\nabla_n \overset{*}{R}{}^a{}_b{}^{kn} + \overset{*}{R}{}^c{}_b{}^{kn} T^a{}_{cn} - T^c{}_{bn} \overset{*}{R}{}^a{}_c{}^{kn} = 0 \tag{128}$$

or, if we drop the matrix indices, as

$$\nabla_n \overset{*}{R}{}^{kn} + \overset{*}{R}{}^{kn} T_n - T_n \overset{*}{R}{}^{kn} = 0. \tag{129}$$

# 6 Structural equations of $A_4$ geometry in the form of expanded, completely geometrized Einstein-Yang-Mills set of equations

Einstein believed that one of the main problems of the unified field theory was the geometrization of the energy-momentum tensor of matter on the right-hand side of his equations. This problem can be solved if we use as the space of events the geometry of absolute parallelism and the structural Cartan equations for this geometry.

In fact, folding the equations $(B)$, written as

$$R^i{}_{jkm} + 2\nabla_{[k} T^i_{|j|m]} + 2T^i_{s[k} T^s_{|j|m]} = 0 \tag{130}$$

in indices i and k, gives

$$R_{jm} = -2\nabla_{[i} T^i_{|j|m]} - 2T^i_{s[i} T^s_{|j|m]}. \tag{131}$$

If then we fold the equations (131) with the metric tensor $g^{jm}$, we have

$$R = -2g^{jm}(\nabla_{[i} T^i_{|j|m]} + 2T^i_{s[i} T^s_{|j|m]}). \tag{132}$$

Forming, using (131) and (132), the Einstein tensor

$$G_{jm} = R_{jm} - \frac{1}{2} g_{jm} R,$$

we obtain the equations

$$R_{jm} - \frac{1}{2} g_{jm} R = \nu T_{jm}, \tag{133}$$

which are similar to Einstein's equations, but with the geometrized right-hand side defined as

$$T_{jm} = -\frac{2}{\nu}\{(\nabla_{[i} T^{i}{}_{|j|m]} + T^{i}{}_{s[i} T^{s}{}_{|j|m]}) - \\ -\frac{1}{2} g_{jm} g^{pn} (\nabla_{[i} T^{i}{}_{|p|n]} + T^{i}{}_{s[i} T^{s}{}_{|p|n]})\} \tag{134}$$

Using the notation

$$P_{jm} = (\nabla_{[i} T^{i}{}_{|j|m]} + T^{i}{}_{s[i} T^{s}{}_{|j|m]})$$

then, by (134), we have

$$T_{jm} = -\frac{2}{\nu}(P_{jm} - \frac{1}{2} g_{jm} g^{pn} P_{pn}). \tag{135}$$

Tensor (135) has parts that are both symmetrical and skew-symmetrical in indices $j$ and $m$, i.e.,

$$T_{jm} = T_{(jm)} + T_{[jm]}. \tag{136}$$

The left-hand side of the equations (133) is always symmetrical in indices $j$ and $m$, therefore these equations can be written as

$$R_{jm} - \frac{1}{2} g_{jm} R = \nu T_{(jm)}, \tag{137}$$

$$T_{[jm]} = \frac{1}{\nu}(-\nabla_i \Omega^{..i}_{jm} - \nabla_m A_j - A_s \Omega^{..s}_{jm}) = 0, \tag{138}$$

where

$$A_j = T^{i}_{ji}. \tag{139}$$

Relationship (138) can be taken to be the equations obeyed by the torsion fields $\Omega^{..i}_{jm}$, which form the energy-momentum tensor (135).

In the case where the field $T^{i}_{jk}$ is skew-symmetrical in all the three indices, we get

$$T_{ijk} = -T_{jik} = T_{jki} = -\Omega_{ijk}. \tag{140}$$

For such fields the equations (138) become simple, namely

$$\nabla_i \Omega^{..i}_{jm} = 0. \tag{141}$$

The energy-momentum tensor (135) is symmetrical in indices $j, m$ and appears to be given by

$$T_{jm} = \frac{1}{\nu}(\Omega^{..i}_{sm} \Omega^{..s}_{ji} - \frac{1}{2} g_{jm} \Omega^{.ji}_{s} \Omega^{..s}_{ji}). \tag{142}$$

By (137), we have

$$T_{jm} = \frac{1}{\nu}(R_{jm} - \frac{1}{2}g_{jm}R). \tag{143}$$

Using (131), (140) and (142) gives

$$R_{jm} = \Omega^{..i}_{sm}\Omega^{..s}_{ji}, \tag{144}$$

$$R = g^{jm}\Omega^{..i}_{sm}\Omega^{..s}_{ji} = \Omega^{.ji}_{s}\Omega^{..s}_{ji}. \tag{145}$$

Substituting (144) and (145) into (143), we arrive at the energy-momentum tensor (142).

Through the field (140) we can define the pseudo-vector $h_m$ as follows

$$\Omega^{ijk} = \varepsilon^{ijkm}h_m, \quad \Omega_{ijk} = \varepsilon_{ijkm}h^m, \tag{146}$$

where $\varepsilon_{ijkm}$ is the fully skew-symmetrical Levi-Chivita symbol.

In terms of the pseudo-vector $h_m$ we can write the tensor (142) as follows

$$T_{jm} = \frac{1}{\nu}(h_j h_m - \frac{1}{2}g_{jm}h^i h_i). \tag{147}$$

Substituting the relationships (146) into (141), we get

$$h_{m,j} - h_{j,m} = 0. \tag{148}$$

These equations have two solutions: the trivial one, where $h_m = 0$, and

$$h_m = \psi_{,m}, \tag{149}$$

where $\Psi$ is a pseudo-scalar.

Writing the energy-momentum tensor (148) in terms of this pseudo-scalar, we will have

$$T_{jm} = \frac{1}{\nu}(\psi_{,j}\psi_{,m} - \frac{1}{2}g_{jm}\psi^{,i}\psi_{,i}). \tag{150}$$

Tensor (150) is the energy-momentum tensor of a pseudo-scalar field.

Let us now decompose the Riemann tensor $R_{ijkm}$ into irreducible parts

$$R_{ijkm} = C_{ijkm} + g_{i[k}R_{m]j} + g_{j[k}R_{m]i} + \frac{1}{3}Rg_{i[m}g_{k]j}, \tag{151}$$

where $C_{ijkm}$ is the Weyl tensor; the second and third terms are the traceless part of the Ricci tensor $R_{jm}$ and $R$ is its trace.

Using the equations (133), written as

$$R_{jm} = \nu\left(T_{jm} - \frac{1}{2}g_{jm}T\right), \tag{152}$$

we will rewrite the relationship (151) as

$$R_{ijkm} = C_{ijkm} + 2\nu g_{[k(i}T_{j)m]} - \frac{1}{3}\nu T g_{i[m}g_{k]j}, \tag{153}$$

where $T$ is the tensor trace (135).

Now we introduce the tensor current

$$J_{ijkm} = 2g_{[k(i}T_{j)m]} - \frac{1}{3}T g_{i[m}g_{k]j} \tag{154}$$

and represent the tensor (153) as the sum

$$R_{ijkm} = C_{ijkm} + \nu J_{ijkm}. \tag{155}$$

Substituting this relationship into the equations (130), we will arrive at

$$C_{ijkm} + 2\nabla_{[k}T_{|ij|m]} + 2T_{is[k}T^{s}_{|j|m]} = -\nu J_{ijkm}. \tag{156}$$

Equations (156) are the Yang-Mills equations with a geometrized source, which is defined by the relationship (154). In equations (156) for the Yang-Mills field we have the Weyl tensor $C_{ijkm}$, and the potentials of the Yang-Mills field are the Ricci rotation coefficients $T^{i}_{jk}$.

We now substitute the relationship (155) into the second Bianchi identities $(D)$

$$\nabla_{[n}R_{|ij|km]} + R^{s}_{j[km}T_{|is|n]} - T^{s}_{j[n}R_{|is|km]} = 0. \tag{157}$$

We thus arrive at the equations of motion

$$\nabla_{[n}C_{|ij|km]} + C^{s}_{j[km}T_{|is|n]} - T^{s}_{j[n}C_{|is|km]} = -\nu J_{nijkm} \tag{158}$$

for the Yang-Mills field $C_{ijkm}$, such that the source $J_{nijkm}$ in them is given in terms of the current (154) as follows:

$$J_{nijkm} = \nabla_{[n}J_{|ij|km]} + J^{s}_{j[km}T_{|is|n]} - T^{s}_{j[n}R_{|is|km]}. \tag{159}$$

Using the geometrized Einstein equations (133) and the Yang-Mills equations (156), we can represent the structural Cartan equations $(A)$ and $(B)$ as an extended set of Einstein-Yang-Mills equations

$$\boxed{\begin{array}{cr} \nabla_{[k}e^{a}_{j]} + T^{i}_{[kj]}e^{a}{}_{i} = 0, & (A) \\ R_{jm} - \frac{1}{2}g_{jm}R = \nu T_{jm}, & (B.1) \\ C^{i}_{jkm} + 2\nabla_{[k}T^{i}_{|j|m]} + 2T^{i}_{s[k}T^{s}_{|j|m]} = -\nu J^{i}{}_{jkm}, & (B.2) \end{array}} \tag{160}$$

in which the geometrized sources $T_{jm}$ and $J_{ijkm}$ are given by (135) and (154).

For the case of Einstein's vacuum the equations (160) are much simpler

$$\boxed{\begin{array}{cr} \nabla_{[k}e^{a}_{j]} + T^{i}_{[kj]}e^{a}{}_{i} = 0, & (i) \\ R_{jm} = 0, & (ii) \\ C^{i}{}_{jkm} + 2\nabla_{[k}T^{i}_{|j|m]} + 2T^{i}_{s[k}T^{s}_{|j|m]} = 0. & (iii) \end{array}} \tag{161}$$

The equations of motion (158) for the Yang-Mills field $C_{ijkm}$ will then become

$$\nabla_{[n}C_{|ij|km]} + C^{s}_{j[km}T_{|is|n]} - T^{s}_{j[n}C_{|is|km]} = 0. \tag{162}$$

Equations $(A)$ and $(B.2)$ can be written in matrix form

$$\nabla_{[k}e^{a}{}_{m]} - e^{b}{}_{[k}T^{a}{}_{|b|m]} = 0, \tag{A}$$

$$C^{a}{}_{bkm} + 2\nabla_{[k}T^{a}{}_{|b|m]} + 2T^{a}{}_{f[k}T^{f}{}_{|b|m]} = -\nu J^{a}{}_{bkm}, \tag{B.2}$$

where the current

$$J^{a}{}_{bkm} = 2g_{[k}{}^{(a}T_{b)m]} - \frac{1}{3}Tg^{a}{}_{[m}g_{k]b}, \tag{163}$$

is given by

$$T^{a}{}_{m} = \frac{1}{\nu}(R^{a}{}_{m} - \frac{1}{2}g^{a}{}_{m}R), \tag{B.1}$$
$$m = 0,1,2,3, \quad a = 0,1,2,3.$$

By writing the equations (158) in matrix form, we have

$$\nabla_{[n}C^{a}{}_{|b|km]} + C^{c}{}_{b[km}T^{a}{}_{|c|n]} - T^{c}{}_{b[n}C^{a}{}_{|a|km]} = -\nu J^{a}{}_{nbkm}, \tag{164}$$

where

$$J^{a}{}_{nbkm} = \nabla_{[n}J^{a}{}_{|b|km]} + J^{c}{}_{b[km}T^{a}{}_{|c|n]} - T^{c}{}_{b[n}J^{a}{}_{|c|km]}. \tag{165}$$

Dropping the matrix indices in the matrix equations, we have

$$\nabla_{[k}e_{m]} - e_{[k}T_{m]} = 0, \tag{A}$$

$$C_{km} + 2\nabla_{[k}T_{m]} - [T_k, T_m] = -\nu J_{km}, \tag{B.2}$$

$$\nabla_n \overset{*}{C}{}^{kn} + [\overset{*}{C}{}^{kn}, T_n] = -\nu \overset{*}{J}{}^{k}, \tag{D}$$

where the dual matrices $\overset{*}{C}{}^{kn}$ and $\overset{*}{J}{}^{k}$ are given by

$$\overset{*}{C}{}^{kn} = \varepsilon^{knij}C_{ij},$$
$$\overset{*}{J}{}^{nk} = \varepsilon^{nkim}J_{im}, \tag{166}$$
$$\overset{*}{J}{}^{k} = \{\nabla_n \overset{*}{J}{}^{kn} + [\overset{*}{J}{}^{kn}, T_n]\}. \tag{167}$$

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