RICCI TORSION AND NP FORMALISM

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1 Three main spinor bases of $\mathbf{A_4}$ geometry

The geometry of absolute parallelism, as laid down in vector basis, enables the structural equations of this geometry of be represented as right (invariant with respect to the T_4^+ and $SO^+(3.1)$) groups and left (invariant with respect to the T_4^- and the $SO^-(3.1)$ groups) groups of the structural equations (A^+), (B^+) and (A^-) and (B^-), respectively. Equations (A^+), (B^+) (or (A^-), (B^-)) can, in turn, be split by a transition into a group of equations, whose component fields have opposite spins. For this purpose, we have to use spinor basis and some elements of spinor analysis.

We will view the spinor geometry A_4 as a differentiable manifold X_4, such that at each point M with the translational coordinates x $(i = 0, 1, 2, 3)$ a two-dimensional spinor space $\mathcal{C}^2$ is introduced [1]. There are three possibilities for introducing the spinor basis in the spinor space $\mathcal{C}^2$:

(a) spinor Γ-basis formed by the Infeld-Van der Werden symbols $\sigma^i_{\alpha\dot\beta}$ [2], which satisfy the equality

$$\nabla_n \sigma^i_{\alpha\dot\beta} = 0; \tag{1}$$

(b) spinor Δ-basis formed by the Newman-Penrose symbols $\sigma^i_{A\dot B}$ [3], which satisfy the equality

$$\stackrel{*}{\nabla}_n \sigma^i_{A\dot B} = 0; \tag{2}$$

(c) spinor dyad basis ξ^α_B, which satisfies the equality [4]

$$\varepsilon^{BD}\xi_{\alpha D}\nabla_k \xi^\alpha_B = 0. \tag{3}$$

In relationships (1)–(3) the indices $\alpha, \dot\beta, \ldots$ and $A, \dot B, \ldots$ are spinor indices that take on the values 0,1 and $\dot 0, \dot 1$. Any local vector A^i that belongs to $\mathcal{C}^2$ can be represented as a spin-tensor of the second rank either in the spinor Γ- basis

$$A^i = A^{\alpha\dot\beta}\sigma^i_{\alpha\dot\beta}, \tag{4}$$

or in the spinor Δ-basis

$$A^i = A^{A\dot B}\sigma^i_{A\dot B}. \tag{5}$$

All the spin-tensors associated with the Γ-basis will have the spinor indices $\alpha, \dot\beta, \ldots$, and the spin-tensors associated with Δ basis will have spinor indices $A, \dot B, \ldots$. As to dyad ξ^α_B, it is a connection between Γ- and Δ-basis

$$\sigma^i_{A\dot B} = \sigma^i_{\alpha\dot\beta}\xi^\alpha_A \bar\xi^{\dot\beta}_{\dot B}. \tag{6}$$

Here

$$\bar{\xi}_B^{\dot{\beta}} = \overline{\xi_B^{\beta}},$$

and the bar on the right-hand side of the equality implies complex conjugation.

Spinor Δ-basis is connected with the vector basis e_i^a by

$$\sigma^i_{\ A\dot{B}} = e^i_a \sigma^a_{\ A\dot{B}}, \tag{7}$$

$$\sigma_i^{A\dot{B}} = e_i^a \sigma_a^{A\dot{B}}, \tag{8}$$

where $\sigma_i^{\ A\dot{B}}$ are complex Hermitian ($\overline{\sigma_i^{\ A\dot{B}}} = \sigma_i^{\ \dot{A}B}$) matrices, and the matrices $\sigma^a_{\ A\dot{B}}$ and $\sigma_a^{\ A\dot{B}}$ have the form

$$\sigma^a_{\ A\dot{B}} = (2)^{-1/2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \tag{9}$$

$$\sigma_a^{A\dot{B}} = (2)^{-1/2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \tag{10}$$

where

$$\det(\sigma^a_{A\dot{B}}) = i, \quad \det(\sigma_a^{A\dot{B}}) = -i.$$

From the orthogonality conditions for the tetrad e_a^i

$$e^a_{\ i} e^j_{\ a} = \delta_i^{\ j}, \quad e^a_{\ i} e^i_{\ b} = \delta^a_{\ b} \tag{11}$$

and the relationships (7)–(10) follows the orthogonality conditions for the spinor Δ-basis

$$\sigma_i^{A\dot{B}} \sigma^j_{\ A\dot{B}} = \delta_i^{\ j}, \tag{12}$$

$$\sigma_i^{A\dot{B}} \sigma^i_{\ C\dot{E}} = \delta^A_{\ C} \delta^{\dot{B}}_{\ \dot{E}}. \tag{13}$$

For the spinor Γ-basis the following orthogonality conditions hold [17]

$$\sigma_i^{\alpha\dot{\beta}} \sigma^j_{\ \alpha\dot{\beta}} = \delta_i^{\ j}, \tag{14}$$

$$\sigma_i^{\alpha\dot{\beta}} \sigma^i_{\ \rho\dot{\nu}} = \delta^\alpha_{\ \rho} \delta^{\dot{\beta}}_{\ \dot{\nu}}. \tag{15}$$

Whence, by (6) and (12)-(13), follow the orthogonality conditions for the spinor dyad

$$\begin{gathered} \xi^o_\alpha \xi^\alpha_1 = 1, \\ \xi^\alpha_o \xi^o_\alpha = -\xi^o_\alpha \xi^\alpha_o = 0, \\ \xi^\alpha_1 \xi^1_\alpha = 0. \end{gathered} \tag{16}$$

In addition, there are the relationships [17]

$$\begin{gathered} \xi^o_\alpha \xi^\beta_o - \xi^1_\alpha \xi^\beta_o = \delta^\beta_\alpha, \\ \xi^o_\alpha \xi^1_\beta - \xi^1_\alpha \xi^o_\beta = \varepsilon_{\alpha\beta}, \end{gathered} \tag{17}$$

where

$$\varepsilon_{\alpha\beta} = \varepsilon^{\alpha\beta} = \varepsilon_{\dot\gamma\dot\delta} = \varepsilon^{\dot\gamma\dot\delta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{18}$$

is the fundamental spinor [3] that obeys the following relationships:

$$\varepsilon_{\alpha\beta}\varepsilon^{\kappa\beta} = \varepsilon_{\alpha}^{\kappa} = -\varepsilon_{\alpha}^{\kappa}, \tag{19}$$

$$\varepsilon_{\alpha\beta}\varepsilon^{\kappa\pi} = \delta_{\alpha}^{\kappa}\delta_{\beta}^{\pi} - \delta_{\alpha}^{\pi}\delta_{\beta}^{\kappa}, \tag{20}$$

$$\varepsilon_{\alpha}^{\alpha} = 2, \tag{21}$$

$$\varepsilon_{\alpha[\beta}\varepsilon_{\kappa\delta]} = 0, \tag{22}$$

$$\varepsilon_{\alpha}^{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{23}$$

The fundamental spinor $\varepsilon_{\alpha\beta}$ increases and decreases the indices on the spin-tensor associated with the Γ-basis, similar to the metric tensor g_{ik} in the vector basis. In the spinor Δ-basis it has the form

$$\varepsilon_{AB} = \varepsilon_{\alpha\beta}\xi_{A}^{\alpha}\xi_{B}^{\beta}, \tag{24}$$

so that

$$\varepsilon^{AB} = \varepsilon_{AB} = \varepsilon^{\dot C\dot D} = \varepsilon_{\dot C\dot D} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{25}$$

The fundamental spinor ε_{AB} increases and decreases indices on the spin-tensors associated with the Δ-basis. For example, we have

$$\begin{array}{ll} \chi^{\dots A\dots}_{\dots\;\dots}\varepsilon_{AB} = \chi^{\dots\;\dots}_{\dots B\dots}, & \varepsilon^{AB}\chi^{\dots\;\dots}_{\dots B\dots} = \chi^{\dots A\dots}_{\dots\;\dots}, \\ \varphi^{\dots\dot A\dots}_{\dots\;\dots}\varepsilon_{\dot A\dot B} = \varphi^{\dots\;\dots}_{\dots\dot B\dots} & \varepsilon^{\dot A\dot B}\varphi^{\dots\;\dots}_{\dots\dot B\dots} = \varphi^{\dots\dot A\dots}_{\dots\;\dots}. \end{array} \tag{26}$$

If the spinor is skew-symmetric in two indices

$$\theta_{\dots A\dots B\dots} = -\theta_{\dots B\dots A\dots}, \tag{27}$$

then, using the fundamental spinor ε_{AB}, it can be represented as [3]

$$\theta_{\dots A\dots B\dots} = \frac{1}{2}\varepsilon_{AB}\theta^{\dots\;\dots C\dots}_{\dots C\dots}. \tag{28}$$

The same properties are valid in the spinor Γ-basis for the fundamental spinor $\varepsilon_{\alpha\beta}$.

2 Spinor representation of the structural Cartan equations of A_4 geometry

The relationship (28) makes it possible to reduce spinors skew-symmetric in primed and unprimed indices to spinors that are completely (or partially) symmetrical in primed and anprimed indices. In the space of spinors of this type irreducible representations of the groups $SL(2.C)$ are realized [5]. This group replaces the group $SO(3.1)$ on passing over to the spinor basis.

Definition 6.1. We will say that the components of a spinor with r symmetrical lower indices and with s symmetrical lower primed indices are transformed in $D(r/2, s/2)$ irreducible representation of the group $SL(2.C)$.

For example, the spinor

$$F_{AB} = F_{BA}$$

is transformed in $D(1.0)$, and the spinor

$$F_{\dot{C}\dot{D}} = F_{\dot{D}\dot{C}}$$

in the $D(0.1)$ irreducible representation of the group $SL(2.C)$.

We will write the main relationships of the A_4 geometry in the spinor Δ-basis. This can be accomplished using the spinor representation of the arbitrary tensor $T^{\dots i \dots \dots}_{\dots \dots j \dots}$ in the Δ-basis

$$T^{\dots A\dot{B} \dots \dots}_{\dots \dots C\dot{E} \dots} = \sigma^{A\dot{B}}_{i} T^{\dots i \dots \dots}_{\dots \dots j \dots} \sigma^{j}_{C\dot{E}} \tag{29}$$

or simply replacing the matrix indices by two spinor ones as follows:

$$e^{a}_{i} \leftrightarrow \sigma^{A\dot{B}}_{i}, \tag{30}$$

$$T^{a}{}_{bm} \leftrightarrow T^{A\dot{B}}{}_{C\dot{D}m}, \tag{31}$$

$$R^{a}{}_{bkm} \leftrightarrow R^{A\dot{B}}{}_{C\dot{D}km}, \tag{32}$$

$$\eta_{ab} \leftrightarrow \eta_{A\dot{B}C\dot{D}} = \varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}}, \tag{33}$$

and so on.

Proposition 6.1. In the spinor Δ-basise the metric tensor g_{ij} of the A_4 geometry has the form

$$g_{ij} = \varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}}\sigma^{A\dot{B}}_{i}\sigma^{C\dot{D}}_{j}. \tag{34}$$

Proof. Substituting into

$$g_{ij} = \eta_{ab} e^{a}{}_{i} e^{b}{}_{j}$$

the relationships (7) and (8) written as

$$e^{a}{}_{i} = \sigma^{A\dot{B}}_{i}\sigma^{a}{}_{A\dot{B}}, \quad e^{b}{}_{j} = \sigma^{C\dot{D}}{}_{j}\sigma^{b}{}_{C\dot{D}}, \tag{35}$$

we have

$$g_{ij} = \eta_{ab}\sigma^{A\dot{B}}_{i}\sigma^{a}_{A\dot{B}}\sigma^{C\dot{D}}_{j}\sigma^{b}_{C\dot{D}}. \tag{36}$$

From the relationships (9), (10), (25) and the definition

$$\eta_{ab} = \eta^{ab} = \mathrm{diag}(1\ -1\ -1\ -1),$$

we obtain the following equality:

$$\eta_{ab}\sigma^a_{A\dot{B}}\sigma^b_{C\dot{D}} = \varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}}.$$

Substituting this into (36), we arrive at the formula (34).

We now write the structural Cartan equations in matrix form

$$\nabla_{[k}e^a_{m]} - e^b_{[k}T^a_{|b|m]} = 0, \qquad (A)$$

$$R^a_{bkm} + 2\nabla_{[k}T^a_{|b|m]} + 2T^a_{c[k}T^c_{|b|m]} = 0. \qquad (B)$$

Using the rules (30)-(32), we write these equations in the spinor Δ-basis

$$\nabla_{[k}\sigma^{A\dot{B}}_{m]} - \sigma^{C\dot{D}}_{[k}T^{A\dot{B}}{}_{|C\dot{D}|m]} = 0, \qquad (37)$$

$$R^{A\dot{B}}{}_{C\dot{D}km} + 2\nabla_{[k}T^{A\dot{B}}{}_{|C\dot{D}|m]} + 2T^{A\dot{B}}{}_{E\dot{F}[k}T^{E\dot{F}}{}_{|C\dot{D}|m]} = 0. \qquad (38)$$

Consequently, the second Bianchi identity of the A_4 geometry

$$\nabla_{[n}R^a_{|b|km]} + R^c_{b[km}T^a_{|c|n]} - T^c_{b[n}R^a_{|c|km]} = 0 \qquad (D)$$

in the spinor Δ-basis becomes

$$\nabla_{[n}R^{A\dot{B}}{}_{|C\dot{D}|km]} + R^{E\dot{F}}{}_{C\dot{D}[km}T^{A\dot{B}}{}_{|C\dot{D}|n]} - T^{E\dot{F}}{}_{C\dot{D}[n}R^{A\dot{E}}{}_{|E\dot{F}|km]} = 0. \qquad (39)$$

Proposition 6.2. If $F_{ij} = -F_{ji}$ is a real skew-symmetrical tensor, then the corresponding spinor

$$F_{A\dot{B}C\dot{D}} = F_{ij}\sigma^i_{A\dot{B}}\sigma^j_{C\dot{D}} \qquad (40)$$

can be represented in the form

$$F_{A\dot{B}C\dot{D}} = \frac{1}{2}(\varepsilon_{\dot{B}\dot{D}}F_{AC} + \varepsilon_{AC}\overline{F}_{\dot{B}\dot{D}}), \qquad (41)$$

where the spinor

$$F_{AC} = F_{CA} \qquad (42)$$

is transformed in the $D(1.0)$ irreducible representation of the group $SL(2.C)$, and the spinor

$$\overline{F}_{BD} = F_{\dot{B}\dot{D}} = F_{\dot{D}\dot{B}} \qquad (43)$$

in the $D(0.1)$ irreducible representation of the same group.

Proof. Since the tensor F_{ij} is skew-symmetric, we have, by (40),

$$F_{A\dot{B}C\dot{D}} = -F_{C\dot{D}A\dot{B}}. \qquad (44)$$

We rewrite this as

$$F_{A\dot{B}C\dot{D}} = \frac{1}{2}(F_{A\dot{B}C\dot{D}} - F_{C\dot{D}A\dot{B}} = \frac{1}{2}(F_{A\dot{B}C\dot{D}} - F_{C\dot{D}A\dot{B}} + F_{C\dot{D}A\dot{B}} - \\ -F_{C\dot{D}A\dot{B}}). \qquad (45)$$

Using the fundamental spinor (25), we can write (45) as follows:

$$F_{A\dot{B}C\dot{D}} = \frac{1}{2}(\varepsilon_{AC}F_{F\dot{B}}{}^{F}{}_{\dot{D}} + \varepsilon_{\dot{B}\dot{D}}F_{A\dot{E}C}{}^{\dot{E}}). \tag{46}$$

Denoting $F_{AC} = (1/2)F_{A\dot{E}C}{}^{\dot{E}}$, we have, by (44),

$$F_{AC} = \frac{1}{2}F_{A\dot{E}C}{}^{\dot{E}} = -\frac{1}{2}F_{A}{}^{\dot{E}}{}_{C\dot{E}} = F_{CA}. \tag{47}$$

Further, introducing the notation $\overline{F}_{\dot{B}\dot{D}} = \frac{1}{2}F_{F\dot{B}}{}^{F}{}_{\dot{D}}$ and considering that F_{ij} is real, we find

$$\overline{F}_{\dot{B}\dot{D}} = \frac{1}{2}F_{F\dot{B}}{}^{F}{}_{\dot{D}} = \frac{1}{2}\overline{F}_{\dot{F}B}{}^{\dot{F}}{}_{D} = \overline{F}_{BD}. \tag{48}$$

Substituting the relationships (47) and (48) into (46), we arrive at (44). By definition, the spinor $F_{AC} = F_{CA}$ belongs to the $D(1.0)$ irreducible representation of the groups $SL(2.C)$; and spinor $\overline{F}_{\dot{B}\dot{D}} = \overline{F}_{\dot{D}\dot{B}}$ – to the $D(0.1)$ irreducible representation of the group.

Since the quantities $T_{A\dot{B}C\dot{D}m}$ and $R_{A\dot{B}C\dot{D}kn}$ in the equations (37) and (38) are skew-symmetric in the pair of spinor matrix indices $A\dot{B}$ and $C\dot{D}$, we can represent them, by (27)–(28), as

$$T_{A\dot{B}C\dot{D}k} = \frac{1}{2}(\varepsilon_{\dot{B}\dot{D}}T_{ACk} + \varepsilon_{AC}T^{+}_{\dot{B}\dot{D}k}), \tag{49}$$

$$R_{A\dot{B}C\dot{D}kn} = \frac{1}{2}(\varepsilon_{\dot{B}\dot{D}}R_{ACkn} + \varepsilon_{AC}R^{+}_{\dot{B}\dot{D}kn}), \tag{50}$$

where

$$T_{ACk} = \frac{1}{2}\varepsilon^{\dot{B}\dot{D}}T_{A\dot{B}C\dot{D}k}, \quad T^{+}_{\dot{B}\dot{D}k} = \frac{1}{2}\varepsilon^{AC}T_{A\dot{B}C\dot{D}k}, \tag{51}$$

$$R_{ACkn} = \frac{1}{2}\varepsilon^{\dot{B}\dot{D}}R_{A\dot{B}C\dot{D}kn}, \quad R^{+}_{\dot{B}\dot{D}kn} = \frac{1}{2}\varepsilon^{AC}R_{A\dot{B}C\dot{D}kn}. \tag{52}$$

In these relationships the + sign with the spinor matrices implies Hermitian conjugation.

3 Splitting of structural Cartan equations into irreducible representations of the group SL(2.C)

Matrices (51) and (52) can be transformed in the spinor indices as follows:

$$T^{A'}_{C'k} = S^{A'}_{A}T^{A}{}_{Ck}S^{C}_{C'} + S^{A'}_{A}S^{A}{}_{C',k}, \tag{53}$$

$$T^{+\dot{B}'}_{\dot{D}'k} = S^{+\dot{B}'}_{\dot{B}}T^{+\dot{B}}_{\dot{D}k}S^{+\dot{D}}_{\dot{D}'} + S^{+\dot{B}'}_{\dot{B}}S^{\dot{B}}{}_{\dot{D}',k}, \tag{54}$$

$$R^{A'}{}_{C'kn} = S^{A'}_{A}R^{A}{}_{Ckn}S^{C}_{C'}, \tag{55}$$

$$R^{+\dot{B}'}{}_{\dot{D}'kn} = S^{+\dot{B}'}_{\dot{B}}R^{+\dot{B}}{}_{\dot{D}kn}S^{+\dot{D}}_{\dot{D}'}. \tag{56}$$

Matrices of the transformations $S^{A'}_{A}$ and $S^{+B}_{\dot{B}}$ form the group $SL(2.C)$, and the matrices

$$S^{A'}_{A} \tag{57}$$

form the subgroup

$$SL^{+}(2.C) \tag{58}$$

of the group $SL(2.C)$, in which the spinors belonging to the irreducible representation $D(r/2,0)$ are transformed.

On the other hand, the matries

$$S^{+\dot{B}'}{}_{\dot{B}} \tag{59}$$

form the subgroup

$$SL_{-}(2.C) \tag{60}$$

of the group $SL(2.C)$, in which the spinors belonging to the irreducible representation $D(0,s/2)$ are transformed. These properties of the spinors enable the structural Cartan equations to be split into equations that contain spinors transformed in $D(r/2,0)$ or $D(0,s/2)$ irreducible representations of the group $SL(2.C)$.

Proposition 6.3. The second structural Cartan equations (B) in the spinor Δ-basis are split into the equations of the form

$$R_{ACkn} + 2\nabla_{[k}T_{|AC|n]} + 2T_{AE[k}T^{E}_{|C|n]} = 0, \tag{61}$$

$$R^{+}_{\dot{B}\dot{D}kn} + 2\nabla_{[k}T^{+}_{|\dot{B}\dot{D}|n]} + 2T^{+}_{\dot{B}\dot{F}[k}T^{+\dot{F}}{}_{|\dot{D}|n]} = 0. \tag{62}$$

Proof. We write the second structural Cartan equations (38) as

$$B_{A\dot{B}C\dot{D}kn} = R_{A\dot{B}C\dot{D}km} + 2\nabla_{[k}T_{|A\dot{B}C\dot{D}|m]} + 2T_{A\dot{B}E\dot{F}[k}T^{E\dot{F}}_{|C\dot{D}|m]} = 0. \tag{63}$$

Using the fact that the spinor $B_{A\dot{B}C\dot{D}kn}$ is skew-symmetric in the pair of spinor indices $A\dot{B}$ and $\dot{C}D$, we will write it in the form

$$B_{A\dot{B}C\dot{D}kn} = \frac{1}{2}(\varepsilon_{\dot{B}\dot{D}}B_{ACkn} + \varepsilon_{AC}B^{+}_{\dot{B}\dot{D}kn}) = 0, \tag{64}$$

where

$$B_{ACkn} = \frac{1}{2}\varepsilon^{\dot{B}\dot{D}}B_{A\dot{B}C\dot{D}kn} = 0, \tag{65}$$

$$B^{+}_{\dot{B}\dot{D}kn} = \frac{1}{2}\varepsilon^{AC}B_{A\dot{B}C\dot{D}kn} = 0. \tag{66}$$

Substituting (61) into the equations (65) and (66) and using the matrices (51) and (52), we will arrive at the structural equations (61) and (62) in split form. In the derivation we have used the properties (19)-(23) of the fundamental spinor ε_{AB}.

Proposition 6.4. Matrices T_{ACk} and $T^{+}_{\dot{B}\dot{D}k}$ in the dyad basis $\xi_{\alpha C}$ have the following form:

$$T_{ACk} = \xi_{\alpha C}\nabla_k\xi^{\alpha}_{A} = T_{ACk}, \tag{67}$$

$$T^{+}_{\dot{B}\dot{D}k} = \bar{\xi}_{\dot{\alpha}\dot{D}}\nabla_k\bar{\xi}^{\dot{\alpha}}_{\dot{B}} = T^{+}_{\dot{B}\dot{D}k}. \tag{68}$$

Proof. We write the matrices

$$T_{abk} = e^{i}{}_{b}\nabla_k e_{ai}$$

in the spinor basis, using the rules (30) and (31)

$$T_{A\dot{B}C\dot{D}k} = \sigma^{i}_{C\dot{D}}\nabla_k\sigma_{A\dot{B}i}. \tag{69}$$

Substituting this expression into the first one of (51) gives

$$T_{ACk} = \frac{1}{2}\varepsilon^{\dot{B}\dot{D}}\sigma^i_{C\dot{D}}\nabla_k\sigma_{A\dot{B}i}. \tag{70}$$

Using the formula (6), we write $\sigma_{A\dot{B}i}$ as

$$\sigma_{A\dot{B}i} = \sigma_{\alpha\dot{\beta}i}\xi^\alpha_A\overline{\xi}^{\dot{\beta}}_{\dot{B}}. \tag{71}$$

Substituting (71) into (70), we have

$$T_{ACk} = \frac{1}{2}\varepsilon^{\dot{B}\dot{D}}\sigma^i_{C\dot{D}}\nabla_k(\sigma_{\alpha\dot{\beta}i}\xi^\alpha_A\overline{\xi}^{\dot{\beta}}_{\dot{B}}) = \frac{1}{2}\varepsilon^{\dot{B}\dot{D}}\sigma^i_{C\dot{D}}\sigma_{\alpha\dot{\beta}i}\nabla_k(\xi^\alpha_A\overline{\xi}^{\dot{\beta}}_{\dot{B}}), \tag{72}$$

since $\nabla_k(\sigma_{\alpha\dot{\beta}i}) = 0$.

Further, considering that

$$\sigma^i_{C\dot{D}}\sigma_{\alpha\dot{\beta}i} = \sigma_{\nu\dot{\gamma}}{}^i\xi^\nu_C\overline{\xi}^{\dot{\gamma}}_{\dot{D}}\sigma_{\alpha\dot{\beta}i} =$$
$$= \delta_{\nu\alpha}\delta_{\dot{\gamma}\dot{\beta}}\xi^\nu_C\overline{\xi}^{\dot{\gamma}}_{\dot{D}} = \xi_{C\alpha}\overline{\xi}_{\dot{D}\dot{\beta}},$$

we will write (72) as

$$T_{ACk} = \frac{1}{2}\varepsilon^{\dot{B}\dot{D}}\xi_{C\alpha}\overline{\xi}_{\dot{D}\dot{\beta}}(\overline{\xi}^{\dot{\beta}}_{\dot{B}}\nabla_k\xi^\alpha_A + \xi^\alpha_A\nabla_k\overline{\xi}^{\dot{\beta}}_{\dot{B}}). \tag{73}$$

In the dyad basis we have the equalities

$$\varepsilon_{\dot{B}\dot{D}} = \overline{\xi}_{\dot{D}\dot{\beta}}\overline{\xi}^{\dot{\beta}}_{\dot{B}}, \quad \varepsilon^{\dot{B}\dot{D}}\overline{\xi}_{\dot{D}\dot{\beta}}\nabla_k\overline{\xi}^{\dot{\beta}}_{\dot{B}} = 0,$$

which are conjugates of (3) and (24). Using these equalities, we can easily obtain (67). Similarly, for the conjugate matrix $T^+{}_{\dot{B}\dot{D}k}$, we have (68).

Proposition 6.5. In the spinor Δ-basis the first structural Cartan equations (A) of the A_4 geometry have the form

$$\nabla_{[k}\sigma^{i]}_{C\dot{D}} - T_{[k|CE}\sigma^E{}_{\dot{D}}|^{i]} - \sigma^{[i}_{|C\dot{F}|}T^+_{k]\dot{D}}{}^{\dot{F}} = 0 \tag{74}$$

or, dropping the matrix indices,

$$\nabla_{[k}\sigma^{i]} - T_{[k}\sigma^{i]} - \sigma^{[i}T^+_{k]} = 0. \tag{75}$$

Proof. Let us take the derivative $\nabla_k\sigma^i_{C\dot{D}}$:

$$\nabla_k\sigma^i_{C\dot{D}} = \nabla_k(\sigma^i_{\alpha\dot{\beta}}\xi^\alpha_C\overline{\xi}^{\dot{\beta}}_{\dot{D}}) = \sigma^i_{\alpha\dot{\beta}}(\overline{\xi}^{\dot{\beta}}_{\dot{D}}\nabla_k\xi^\alpha_C + \xi^\alpha_C\nabla_k\overline{\xi}^{\dot{\beta}}_{\dot{D}}).$$

Using (67) and (68), we will write this relationship as

$$\nabla_k\sigma^i_{C\dot{D}} = \sigma^i_{\alpha\dot{\beta}}(T_{CEk}\xi^{\alpha E}\overline{\xi}^{\dot{\beta}}_{\dot{D}} + T^+_{\dot{D}\dot{F}k}\xi^\alpha_C\overline{\xi}^{\dot{\beta}\dot{F}}). \tag{76}$$

Here we have used the normalization conditions

$$\xi_{\alpha E}\xi^{\alpha E} = 1, \quad \overline{\xi}_{\dot{\beta}\dot{F}}\overline{\xi}^{\dot{\beta}\dot{F}} = 1.$$

Multiplying the terms on the right-hand side (76) we obtain, from (71),

$$\nabla_k \sigma^i_{C\dot{D}} - T_{CEk}\sigma^{iE}_{\dot{D}} - \sigma^{i\dot{F}}_{C} T^{+}_{\dot{D}\dot{F}k} = 0 \tag{77}$$

or

$$\nabla_k \sigma^i_{C\dot{D}} - T_{kCE}\sigma^{Ei}_{\dot{D}} - \sigma^i_{C\dot{F}} T^{+\dot{F}}_{k\dot{D}}. \tag{78}$$

Alternating this relationship in the indices k and i, we obtain the equations (74).

Proposition 6.6. The second Bianchi (D) identities of the A_4 geometry in the spinor Δ-basis are split into the following equations:

$$\nabla^n \overset{*}{R}_{ACkn} - \overset{*}{R}_{ECkn}\, T^E{}_A{}^n + \overset{*}{R}_{EAkn}\, T^E{}_C{}^n = 0, \tag{79}$$

$$\nabla^n \overset{*}{R}{}^{+}_{\dot{B}\dot{D}kn} - \overset{*}{R}{}^{+}_{\dot{F}\dot{D}kn} T^{+\dot{F}n}_{\dot{B}} + \overset{*}{R}{}^{+}_{\dot{F}\dot{B}kn} T^{+\dot{F}n}_{\dot{D}} = 0. \tag{80}$$

Proof. Increasing and decreasing, using the metric tensors η_{ab} and g_{ik}, the tensor indices in the identities (150), we will write them in the form

$$\nabla^n \overset{*}{R}_{abkn} - \overset{*}{R}_{cbkn}\, T^c{}_a{}^n + \overset{*}{R}_{ackn}\, T^c{}_b{}^n = 0. \tag{81}$$

In this equality we now pass over to the spinor indices using (31) and (32) go get

$$\nabla^n \overset{*}{R}_{A\dot{B}C\dot{D}kn} - \overset{*}{R}_{E\dot{F}C\dot{D}kn}\, T^{\dot{E}F}{}_{A\dot{B}}{}^n + \overset{*}{R}_{E\dot{F}A\dot{B}kn}\, T^{\dot{E}F}{}_{C\dot{D}}{}^n = 0. \tag{82}$$

We now write this relationship in the form

$$D^n_{A\dot{B}C\dot{D}kn} = 0, \tag{83}$$

where by $D^n_{A\dot{B}C\dot{D}kn}$ we have denoted all the terms on the left-hand side of (82). Since the relationship (83) are skew-symmetrical in the pair of indices $A\dot{B}$ and $C\dot{D}$, we will write it in the form

$$D^n_{A\dot{B}C\dot{D}kn} = \frac{1}{2}(\varepsilon_{\dot{B}\dot{D}} D^n_{ACkn} + \varepsilon_{AC} D^{+n}{}_{\dot{B}\dot{D}kn}) = 0, \tag{84}$$

where

$$D^n_{ACkn} = \frac{1}{2}\varepsilon^{\dot{B}\dot{D}} D^n_{A\dot{B}C\dot{D}kn} = 0, \quad D^{+n}_{\dot{B}\dot{D}kn} = \frac{1}{2}\varepsilon^{AC} D^n_{A\dot{B}C\dot{D}kn} = 0.$$

Substituting here (82), we will get (79) and (80).

Physically, the spinor splitting of the structural Cartan equations (A) and (B) implies splitting into the equations of "matter " and "skew-symmetry", just as it has been done by Dirac in his derivation of equations for the electron and the positron. We can now write equations that are transformed in the groups $SL^+(2.C)$ as

$$\nabla_{[k}\sigma^{i]}_{C\dot{D}} - T_{[k|CE}\sigma^{E}_{\dot{D}}{}^{|i]} - \sigma^{[i}{}_{|C\dot{F}|} T^{+}_{k]\dot{D}}{}^{\dot{F}} = 0, \tag{A^s}$$

$$R_{ACkn} + 2\nabla_{[k}T_{|AC|n]} + 2T_{AE[k}T^{E}_{|C|n]} = 0, \tag{B^{s+}}$$

and in the group $SL^-(2.C)$ as

$$\nabla_{[k}\sigma^{i]}{}_{C\dot{D}} - T_{[k|CE}\sigma^{E}_{\dot{D}}{}^{|i]} - \sigma^{[i}{}_{|C\dot{F}|} T^{+}_{k]\dot{D}}{}^{\dot{F}} = 0, \tag{A^s}$$

$$R^{+}_{\dot{B}\dot{D}kn} + 2\nabla_{[k}T^{+}_{|\dot{B}\dot{D}|n]} + 2T^{+}_{\dot{B}\dot{F}[k}T^{+\dot{F}}_{|\dot{D}|n]} = 0. \qquad (B^{s-})$$

In the numerations of these formulas s implies transformation in a spinor group. Dropping the matrix indices, we will write these relationships as

$$\nabla_{[k}\sigma^{i]} - T_{[k}\sigma^{i]} - \sigma^{[i}T^{+}_{k]} = 0, \qquad (A^{s})$$

$$R_{kn} + 2\nabla_{[k}T_{n]} - [T_k, T_n] = 0, \qquad (B^{s+})$$

$$\nabla_{[k}\sigma^{i]} - T_{[k}\sigma^{i]} - \sigma^{[i}T^{+}_{k]} = 0, \qquad (A^{s})$$

$$R^{+}_{kn} + 2\nabla_{[k}T^{+}_{n]} - [T^{+}_k, T^{+}_n] = 0. \qquad (B^{s-})$$

Correspondingly, discarding the matrix indices in the equations (79) and (80), we obtain

$$\nabla^n \overset{*}{R}_{kn} + [\overset{*}{R}_{kn}, T^n] = 0, \qquad (D^{s+})$$

$$\nabla^n \overset{*}{R}{}^{+}_{kn} + [\overset{*}{R}{}^{+}_{kn}, T^{+n}] = 0. \qquad (D^{s-})$$

4 Carmeli matrices

Equalities (67) and (68) can be written in matrix form

$$T_k = \xi\nabla_k\xi, \qquad (85)$$

$$T^{+}_k = \xi^{+}\nabla_k\xi^{+}, \qquad (86)$$

where T_k and ξ are 2×2 complex matrices with elements $T^{A}{}_{Bk}$ and ξ^{a}_{A}, respectively. Multiplying T_k by $\sigma^{k}{}_{A\dot{B}}$, we can introduce the traceless Carmeli 2×2 matrices [7]-[9]

$$T_{A\dot{B}} = \sigma^{k}_{A\dot{B}}T_k, \qquad (87)$$

$$A, C\ldots = 0,\ 1, \quad \dot{B}, \dot{D}\ldots = \dot{0},\ \dot{1}$$

with the components

$$T_{0\dot{0}} = \begin{pmatrix} \varepsilon & -\kappa \\ \pi & -\varepsilon \end{pmatrix}, \quad T_{0\dot{1}} = \begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix},$$
$$T_{1\dot{0}} = \begin{pmatrix} \alpha & -\rho \\ \lambda & -\alpha \end{pmatrix}, \quad T_{1\dot{1}} = \begin{pmatrix} \gamma & -\tau \\ \nu & -\gamma \end{pmatrix}. \qquad (88)$$

Using matrices (87), we can define the matrix elements

$(T_{A\dot{B}})_C{}^D =$ (89)

$A\dot{B}$ \ CD	00	01	10	11
$0\dot{0}$	ε	$-\kappa$	π	$-\varepsilon$
$0\dot{1}$	β	$-\sigma$	μ	$-\beta$
$1\dot{0}$	α	$-\rho$	λ	$-\alpha$
$1\dot{1}$	γ	$-\tau$	ν	$-\gamma$

where $(T_{A\dot{B}})_C{}^D$ is the CD element of the matrices $T_{A\dot{B}}$. Consequently, the complex conjugate matrices $T^+{}_{\dot{A}B}$ are

$$(T^+_{\dot{A}B})^{\dot{D}}{}_{\dot{C}} = \begin{array}{c|cccc} & & \dot{C}\dot{D} & & \\ \cline{2-5} \dot{A}B & \dot{0}\dot{0} & \dot{0}\dot{1} & \dot{1}\dot{0} & \dot{1}\dot{1} \\ \hline \dot{0}0 & \overline{\varepsilon} & -\overline{\kappa} & \overline{\pi} & -\overline{\varepsilon} \\ \dot{0}1 & \overline{\beta} & -\overline{\sigma} & \overline{\mu} & -\overline{\beta} \\ \dot{1}0 & \overline{\alpha} & -\overline{\rho} & \overline{\lambda} & -\overline{\alpha} \\ \dot{1}1 & \overline{\gamma} & -\overline{\tau} & \overline{\nu} & -\overline{\gamma} \end{array} . \tag{90}$$

Proposition 6.7. In the Carmeli matrices the first structural Cartan equations (A) of the A_4 geometry have the form

$$\partial_{C\dot{D}}\sigma^i_{A\dot{B}} - \partial_{A\dot{B}}\sigma^i_{C\dot{D}} = (T_{C\dot{D}})_A{}^P\sigma^i_{P\dot{B}} + \sigma^i_{A\dot{R}}(T^+_{\dot{D}C})^{\dot{R}}{}_{\dot{B}} - \\ -(T_{A\dot{B}})_C{}^P\sigma^i_{P\dot{D}} - \sigma^i_{C\dot{R}}(T^+_{\dot{B}A})^{\dot{R}}{}_{\dot{D}}. \tag{91}$$

Proof. We will write the equations (75) as

$$\nabla_k\sigma^i_{C\dot{D}} - \nabla_k\sigma^i_{A\dot{B}} = T_C{}^E{}_k\sigma^i_{\dot{D}E} + \sigma^i_{C\dot{F}}T^{+\dot{F}}_{\dot{D}}{}_k - \\ -T_A{}^C{}_k\sigma^i_{C\dot{B}} - \sigma^i_{A\dot{E}}T^{+\dot{E}}_{\dot{B}\,k}. \tag{92}$$

It is easily seen that the equations (92) represent the difference of the two relationships

$$\nabla_k\sigma^i_{C\dot{D}} = T_C{}^E{}_k\sigma^i_{\dot{D}E} + \sigma^i_{C\dot{F}}T^{+\dot{F}}_{\dot{D}}{}_k, \tag{93}$$

$$\nabla_k\sigma^i_{A\dot{B}} = T_A{}^C{}_k\sigma^i_{C\dot{B}} + \sigma^i_{A\dot{E}}T^{+\dot{E}}_{\dot{B}}{}_k. \tag{94}$$

Multiplying (93) by $\sigma^k{}_{A\dot{B}}$, and (94) by $\sigma^k{}_{C\dot{D}}$, we get

$$\nabla_k\sigma^i_{C\dot{D}}\sigma^k_{A\dot{B}} = T_C{}^E{}_k\sigma^i_{\dot{D}E}\sigma^k_{A\dot{B}} + \sigma^i_{C\dot{F}}T^{+\dot{F}}_{\dot{D}}{}_k\sigma^k_{A\dot{B}}, \tag{95}$$

$$\nabla_k\sigma^i_{A\dot{B}}\sigma^k_{C\dot{D}} = T_A{}^P{}_k\sigma^i_{P\dot{B}}\sigma^k_{C\dot{D}} + \sigma^i_{A\dot{E}}T^{+\dot{E}}_{\dot{B}}{}_k\sigma^k_{C\dot{D}}. \tag{96}$$

We now introduce the notation

$$(T_{A\dot{B}})_C{}^E = T_C{}^E{}_k\sigma^k_{A\dot{B}} \tag{97}$$

and

$$\partial_{A\dot{B}} = \sigma^k_{A\dot{B}}\nabla_k, \tag{98}$$

and rewrite the relationships (95) and (96) as

$$\partial_{C\dot{D}}\sigma^i_{A\dot{B}} = (T_{C\dot{D}})_A{}^P\sigma^i_{P\dot{B}} + \sigma^i_{A\dot{R}}(T^+_{\dot{D}C})^{\dot{R}}{}_{\dot{B}}, \tag{99}$$

$$\partial_{A\dot{B}}\sigma^i_{C\dot{D}} = (T_{A\dot{B}})_C{}^P\sigma^i_{P\dot{D}} + \sigma^i_{C\dot{R}}(T^+_{\dot{B}A})^{\dot{R}}{}_{\dot{D}}. \tag{100}$$

Subtracting from (99) the equality (100), we will arrive at the first structural Cartan equations (91) of the A_4 geometry, written in terms of Carmeli matrices.

Consider now the second structural Cartan equations (B^{s+}), written in matrix forms

$$R_{kn} + 2\nabla_{[k}T_{n]} - [T_k, T_n] = 0. \tag{101}$$

Multiplying the quantity R_{kn} by $\sigma^k{}_{A\dot{B}}$ and $\sigma^n{}_{C\dot{D}}$, we will introduce the traceless Carmeli matrix

$$R_{A\dot{B}C\dot{D}} = R_{kn}\sigma^k_{A\dot{B}}\sigma^n_{C\dot{D}} \tag{102}$$

with the components [44-46]

$$\begin{gathered}
R_{0\dot{1}0\dot{0}} = \begin{pmatrix} \Psi_1 & -\Psi_0 \\ \Psi_2 + 2\Lambda & -\Psi_1 \end{pmatrix}, \quad R_{1\dot{0}0\dot{0}} = \begin{pmatrix} \Phi_{10} & -\Phi_{00} \\ \Phi_{20} & -\Phi_{10} \end{pmatrix}, \\
R_{1\dot{1}1\dot{0}} = \begin{pmatrix} \Psi_3 & -\Psi_2 - 2\Lambda \\ \Psi_4 & -\Psi_3 \end{pmatrix}, \quad R_{1\dot{1}0\dot{1}} = \begin{pmatrix} \Phi_{12} & -\Phi_{02} \\ \Phi_{22} & -\Phi_{12} \end{pmatrix}, \\
R_{1\dot{1}0\dot{0}} = \begin{pmatrix} \Psi_2 + \Phi_{11} - \Lambda & -\Psi_1 - \Phi_{01} \\ \Psi_3 + \Phi_{21} & -\Psi_2 - \Phi_{11} + \Lambda \end{pmatrix}, \\
R_{1\dot{0}0\dot{1}} = \begin{pmatrix} -\Psi_2 + \Phi_{11} + \Lambda & \Psi_1 - \Phi_{01} \\ -\Psi_3 + \Phi_{21} & \Psi_2 - \Phi_{11} - \Lambda \end{pmatrix}.
\end{gathered} \tag{103}$$

Proposition 6.8. In terms of Carmeli spinor matrices (87) and (102), the second structural Cartan equations (B^{s+}) of the A_4 geometry become

$$\begin{aligned}
R_{A\dot{B}C\dot{D}} = \partial_{C\dot{D}}T_{A\dot{B}} - \partial_{A\dot{B}}T_{C\dot{D}} - (T_{C\dot{D}})_A{}^F T_{F\dot{B}} - (T^+_{\dot{D}C})^{\dot{F}}{}_{\dot{B}}T_{A\dot{F}} + \\
+(T_{A\dot{B}})_C{}^F T_{F\dot{D}} + (T^+_{\dot{B}A})^{\dot{F}}{}_{\dot{D}}T_{C\dot{F}} + [T_{A\dot{B}}, T_{C\dot{D}}].
\end{aligned} \tag{104}$$

Proof. We write the equations (101) as

$$R_{kn} = 2\nabla_{[n}T_{k]} + [T_k, T_n] \tag{105}$$

or

$$R_{kn} = \nabla_n T_k - \nabla_k T_n + T_k T_n - T_n T_k. \tag{106}$$

Multiplying this by $\sigma^k{}_{A\dot{B}}\sigma^n{}_{C\dot{D}}$, we will have

$$\begin{aligned}
R_{A\dot{B}C\dot{D}} = \partial_{C\dot{D}}T_k\sigma^k_{A\dot{B}} - \partial_{A\dot{B}}T_n\sigma^n_{C\dot{D}} + T_{A\dot{B}}T_{C\dot{D}} - T_{C\dot{D}}T_{A\dot{B}} = \\
= \partial_{C\dot{D}}T_{A\dot{B}} - \partial_{AB}T_{C\dot{D}} - (\partial_{C\dot{D}}\sigma_{A\dot{B}}{}^k - \partial_{A\dot{B}}\sigma_{C\dot{D}}{}^k)T_k + \\
+[T_{A\dot{B}}, T_{C\dot{D}}].
\end{aligned} \tag{107}$$

We have used here the condition that

$$\sigma_k{}^{A\dot{B}}\sigma^k_{C\dot{E}} = \delta^A_C\delta^{\dot{B}}_{\dot{E}} \tag{108}$$

and the notation

$$\partial_{A\dot{B}} = \sigma^k_{A\dot{B}}\nabla_k. \tag{109}$$

If now in (107) we use the relationships (99) and (100), we will get the equations (103).

Let us write the second Bianchi identities (D^{s+}) of the A_4 geometry in matrix form

$$\nabla^n \overset{*}{R}_{kn} + [\overset{*}{R}_{kn}, T^n] = 0. \tag{110}$$

Multiplying these equations by ${\sigma^n}_{E\dot F}$, we will render them in terms of Carmeli matrices as follows:

$$\begin{aligned}\partial^{C\dot D}\ \overset{*}{R}_{E\dot F C\dot D} &+\sigma^n_{\dot E F}(\nabla^{C\dot D}\sigma^{A\dot B}_n)\ \overset{*}{R}_{A\dot B C\dot D}+\\ +(\nabla_k\sigma^{kC\dot D})\ \overset{*}{R}_{E\dot F C\dot D} &-[T^{C\dot D},\overset{*}{R}_{E\dot F C\dot D}]=0.\end{aligned} \tag{111}$$

Using the relationship (99), we can rewrite the identities (111) as

$$\begin{aligned}\partial^{C\dot D}\ \overset{*}{R}_{E\dot F C\dot D} &-(T^{C\dot D})^A{}_E\ \overset{*}{R}_{A\dot F C\dot D}-\\ -(T^{+\dot D C})_{\dot F}{}^{\dot B}\ \overset{*}{R}_{E\dot B C\dot D} &+(T_P{}^{\dot D})^{CP}\ \overset{*}{R}_{E\dot F C\dot D}+\\ +(T^{+C}_{\dot Q})^{\dot Q\dot D}\ \overset{*}{R}_{E\dot F C\dot D} &+[T^{C\dot D},\overset{*}{R}_{E\dot F C\dot D}]=0.\end{aligned} \tag{112}$$

5 Component-by-component rendering of structural equations of $\mathbf{A}_4$ geometry

Let us now write the equations (91) component by component. For convenience, we will introduce the following notation:

$$\begin{aligned}A^i_{C\dot D A\dot B}=\partial_{C\dot D}\sigma^i_{A\dot B}-\partial_{A\dot B}\sigma^i_{C\dot D}=(T_{C\dot D})_A{}^P\sigma^i_{P\dot B}+\\ +\sigma^i_{A\dot R}(T^+_{\dot D C})^{\dot R}{}_{\dot B}-(T_{A\dot B})_C{}^P\sigma^i_{P\dot D}-\sigma^i_{C\dot R}(T^+_{\dot B A})^{\dot R}{}_{\dot D}.\end{aligned} \tag{113}$$

Also, we will denote the components of the spinor derivative as

$$\partial_{A\dot B}=\begin{array}{c|cc} & \multicolumn{2}{c}{\dot B}\\ \hline A & \dot 0 & \dot 1\\ \hline 0 & D & \delta\\ 1 & \overline{\delta} & \Delta\end{array}, \tag{114}$$

and the components of the spinor Δ-basis as

$$\sigma^i_{A\dot B}=\begin{array}{c|cc} & \multicolumn{2}{c}{\dot B}\\ \hline A & \dot 0 & \dot 1\\ \hline 0 & l^i=(Y^0,V,Y^2,Y^3) & m^i=(\xi^0,\omega,\xi^2,\xi^3)\\ 1 & \overline{m}^i=(\overline{\xi}^0,\overline{\omega},\overline{\xi}^2,\overline{\xi}^3) & n^i=(X^o,U,X^2,X^3)\end{array}. \tag{115}$$

From (113), the spinor component ${A^i}_{0\dot 0 0\dot 1}$ will be

$$\begin{aligned}A^i_{0\dot 0 0\dot 1}=\partial_{0\dot 0}\sigma^i_{0\dot 1}-\partial_{0\dot 1}\sigma^i_{0\dot 0}=(T_{0\dot 0})_0{}^P\sigma^i_{P\dot 1}+\sigma^i_{0\dot R}(T^+_{\dot 0 0})^{\dot R}{}_{\dot 1}-\\ -(T_{0\dot 1})_0{}^P\sigma^i_{P\dot 0}-\sigma^i_{0\dot R}(T^+_{\dot 1 0})^{\dot R}{}_{\dot 0}\end{aligned} \tag{116}$$

or

$$\begin{aligned}A^i_{0\dot 0 0\dot 1}=\partial_{0\dot 0}\sigma^i_{0\dot 1}-\partial_{0\dot 1}\sigma^i_{0\dot 0}=\Big((T_{0\dot 0})_0{}^0\sigma^i_{0\dot 1}+(T_{0\dot 0})_0{}^1\sigma^i_{1\dot 1}\Big)+\\ +\Big(\sigma^i_{0\dot 0}(T^+_{\dot 0 0})^{\dot 0}{}_{\dot 1}+\sigma^i_{0\dot 1}(T^+_{\dot 0 0})^{\dot 1}{}_{\dot 1}\Big)-\Big((T_{0\dot 1})_0{}^0\sigma^i_{0\dot 0}+(T_{0\dot 1})_0{}^1\sigma^i_{1\dot 0}\Big)-\\ -\Big(\sigma^i_{0\dot 0}(T^+_{\dot 1 0})^{\dot 0}{}_{\dot 0}+\sigma^i_{0\dot 1}(T^+_{\dot 1 0})^{\dot 1}{}_{\dot 0}\Big).\end{aligned} \tag{117}$$

Using the notation of (89)-(90) and (114)-(115) for the components $(T_{C\dot{D}})_A{}^P, (T^+_{\dot{B}A})^{\dot{R}}{}_{\dot{D}}, \partial_{A\dot{B}}$ and $\sigma^i_{A\dot{B}}$, we will obtain, by (117),

$$\begin{aligned} Dm^i - \delta l^i = \Big(\varepsilon m^i + (-\kappa)n^i\Big) + \Big(l^i\overline{\pi} + m^i(-\overline{\varepsilon})\Big) - \\ - \Big(\beta l^i + (-\sigma)\overline{m}^i\Big) - \Big(l^i\overline{\alpha} + m^i(-\overline{\rho})\Big) = \\ = -(\overline{\alpha} + \beta - \overline{\pi})l^i - \kappa n^i + \sigma\overline{m}^i + (\overline{\rho} + \varepsilon - \overline{\varepsilon})m^i. \end{aligned} \tag{118}$$

Since the vectors m^i and l^i have the following components:

$$l^i = (Y^0, V, Y^2, Y^3), \quad m^i = (\xi^0, \omega, \xi^2, \xi^3),$$

it follows from (118) that

$$\delta V - D\omega = (\overline{\alpha} + \beta - \overline{\pi})V + \kappa U - \sigma\overline{\omega} - (\overline{\rho} + \varepsilon - \overline{\varepsilon})\omega, \tag{119}$$

$$\delta Y^\alpha - D\xi^\alpha = (\overline{\alpha} + \beta - \overline{\pi})Y^\alpha + \kappa X^\alpha - \alpha\overline{\xi}^\alpha - (\overline{\rho} + \varepsilon - \overline{\varepsilon})\xi^\alpha, \tag{120}$$

$$\alpha = 0, 2, 3.$$

In a similar manner we find the following component rendering of the first structural Cartan equations of the A_4 geometry

$$\delta V - D\omega = (\overline{\alpha} + \beta - \overline{\pi})V + \kappa U - \sigma\overline{\omega} - (\overline{\rho} + \varepsilon - \overline{\varepsilon})\omega, \tag{A.1}$$

$$\delta Y^\alpha - D\xi^\alpha = (\overline{\alpha} + \beta - \overline{\pi})Y^\alpha + \kappa X^\alpha - \sigma\overline{\xi}^\alpha - (\overline{\rho} + \varepsilon - \overline{\varepsilon})\xi^\alpha, \tag{A.2}$$

$$\Delta Y^\alpha - DX^\alpha = (\gamma + \overline{\gamma})Y^\alpha + (\varepsilon + \overline{\varepsilon})X^\alpha - (\tau + \overline{\pi})\xi^\alpha - (\overline{\tau} + \pi)\omega, \tag{A.3}$$

$$\Delta V - DV = (\gamma + \overline{\gamma})V + (\varepsilon + \overline{\varepsilon})U - (\tau + \overline{\pi})\overline{\omega} - (\overline{\tau} + \pi)\omega, \tag{A.4}$$

$$\delta U - \Delta\omega = -\overline{\nu}V + (\tau - \overline{\alpha} - \beta)U + \overline{\lambda}\overline{\omega} + (\mu - \gamma + \overline{\gamma})\omega, \tag{A.5}$$

$$\delta X^\alpha - \Delta\xi^\alpha = -\overline{\nu}Y^\alpha + (\tau - \overline{\alpha} - \beta)X^\alpha + \overline{\lambda\xi}^\alpha + (\mu - \gamma + \overline{\gamma})\xi^\alpha, \tag{A.6}$$

$$\overline{\delta}\omega - \delta\overline{\omega} = (\overline{\mu} - \mu)V + (\overline{\rho} - \rho)U - (\overline{\alpha} - \beta)\overline{\omega} - (\overline{\beta} - \alpha)\omega, \tag{A.7}$$

$$\overline{\delta}\xi^\alpha - \delta\overline{\xi}^\alpha = (\overline{\mu} - \mu)Y^\alpha + (\overline{\rho} - \rho)X^\alpha - (\overline{\alpha} - \beta)\overline{\xi}^\alpha - (\overline{\beta} - \alpha)\xi^\alpha, \tag{A.8}$$

$$\alpha = 0, 2, 3,$$

and the complex conjugate equations $(\bar{A}.1) - (\bar{A}.8)$ (all in all 24 independent equations).

Let us now look at the equations (107) and write them componentwise. For instance, we will derive the $R_{0\mathrm{i}0\dot{0}}$ component of these equations

$$\begin{aligned} R_{0\mathrm{i}0\dot{0}} \;=\; & \partial_{0\dot{0}}T_{0\mathrm{i}} - \partial_{0\mathrm{i}}T_{0\dot{0}} - (T_{0\dot{0}})_0{}^0T_{0\mathrm{i}} - (T_{0\dot{0}})_0{}^1T_{1\mathrm{i}} + \\ & +(T^+_{0\dot{0}})^{\dot{0}}{}_{\mathrm{i}}T_{0\dot{0}} - (T^+_{0\dot{0}})^{\dot{1}}{}_{\mathrm{i}}T_{0\mathrm{i}} + (T_{0\mathrm{i}})_0{}^0T_{0\dot{0}} + (T_{0\mathrm{i}})_0{}^1T_{1\dot{0}} + \\ & +(T^+_{1\dot{0}})^{\dot{0}}{}_{\dot{0}}T_{0\dot{0}} + (T^+_{1\dot{0}})^{\dot{1}}{}_{\dot{0}}T_{0\mathrm{i}} + T_{0\mathrm{i}}T_{0\dot{0}} - T_{0\dot{0}}T_{1\dot{0}}. \end{aligned} \tag{121}$$

Using the matrices (89)-(90), (103) and the spinor derivative (114), we can represent (121) as

$$
\begin{aligned}
\begin{pmatrix} \Psi_1 & -\Psi_0 \\ \Psi_2+2\Lambda & -\Psi_1 \end{pmatrix} &= D\begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix} -\delta\begin{pmatrix} \varepsilon & -\kappa \\ \pi & -\varepsilon \end{pmatrix}- \\
&-\varepsilon\begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix} +\kappa\begin{pmatrix} \gamma & -\tau \\ \nu & -\gamma \end{pmatrix} -\overline{\pi}\begin{pmatrix} \varepsilon & -\kappa \\ \pi & -\varepsilon \end{pmatrix}+ \\
&+\overline{\varepsilon}\begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix} +\beta\begin{pmatrix} \varepsilon & -\kappa \\ \pi & -\varepsilon \end{pmatrix} -\sigma\begin{pmatrix} \alpha & -\rho \\ \lambda & -\alpha \end{pmatrix}+ \\
&+\overline{\alpha}\begin{pmatrix} \varepsilon & -\kappa \\ \pi & -\varepsilon \end{pmatrix} -\overline{\rho}\begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix}+ \\
&+\begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix}\begin{pmatrix} \varepsilon & -\kappa \\ \pi & -\varepsilon \end{pmatrix}-\begin{pmatrix} \varepsilon & -\kappa \\ \pi & -\varepsilon \end{pmatrix}\begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix}.
\end{aligned}
\tag{122}
$$

These equations split into the following three independent equations:

$$
\begin{aligned}
(D-\overline{\rho}+\overline{\varepsilon})\beta-(\delta-\overline{\alpha}+\overline{\pi})\varepsilon-(\alpha+\pi)\sigma+(\mu+\gamma)\kappa-\Psi_1=0, \\
(D-\rho-\overline{\rho}-3\varepsilon+\overline{\varepsilon})\sigma-(\delta-\tau+\overline{\pi}-\overline{\alpha}-3\beta)\kappa-\Psi_0=0, \\
(D-\overline{\rho}+\varepsilon+\overline{\varepsilon})\mu-(\delta+\overline{\pi}-\overline{\alpha}+\beta)\pi-\sigma\lambda+\nu\kappa-2\Lambda-\Psi_2=0.
\end{aligned}
$$

Similarly, we will obtain the following independent equations (B^{s+}):

$$
\begin{gathered}
(D-\rho-\varepsilon-\overline{\varepsilon})\rho-(\overline{\delta}-3\alpha-\overline{\beta}+\pi)\kappa- \\
-\sigma\overline{\sigma}+\tau\overline{\kappa}-\Phi_{00}=0,
\end{gathered}
\tag{$B^{s+}.1$}
$$

$$
\begin{gathered}
(D-\rho-\overline{\rho}-3\varepsilon+\overline{\varepsilon})\sigma-(\delta-\tau+\overline{\pi}-\overline{\alpha}-3\beta)\kappa- \\
-\Psi_0=0,
\end{gathered}
\tag{$B^{s+}.2$}
$$

$$
\begin{gathered}
(D-\rho-\varepsilon+\overline{\varepsilon})\tau-(\Delta-3\gamma-\overline{\gamma})\kappa-\rho\overline{\pi}-\sigma\overline{\tau}-\pi\sigma- \\
-\Psi_1-\Phi_{10}=0,
\end{gathered}
\tag{$B^{s+}.3$}
$$

$$
\begin{gathered}
(D-\rho-\overline{\varepsilon}+2\varepsilon)\alpha-(\overline{\delta}-\overline{\beta}+\pi)\varepsilon-\beta\overline{\sigma}+\kappa\lambda+\overline{\kappa}\gamma- \\
-\pi\rho-\Phi_{10}=0,
\end{gathered}
\tag{$B^{s+}.4$}
$$

$$
\begin{gathered}
(D+\varepsilon+\overline{\varepsilon})\gamma-(\Delta-\gamma-\overline{\gamma})\varepsilon-(\tau+\overline{\pi})\alpha-(\pi+\overline{\tau})\beta- \\
-\pi\tau+\nu\kappa+\Lambda-\Psi_2-\Phi_{11}=0,
\end{gathered}
\tag{$B^{s+}.5$}
$$

$$
(D-\rho+3\varepsilon-\overline{\varepsilon})\lambda-(\overline{\delta}+\pi+\alpha-\overline{\beta})\pi-\mu\overline{\sigma}+\nu\overline{\kappa}-\Phi_{20}=0,
\tag{$B^{s+}.6$}
$$

$$
\begin{gathered}
(D-\overline{\rho}+\overline{\varepsilon})\beta-(\delta-\overline{\alpha}+\overline{\pi})\varepsilon-(\alpha+\pi)\sigma+(\mu+\gamma)\kappa- \\
-\Psi_1=0,
\end{gathered}
\tag{$B^{s+}.7$}
$$

$$
\begin{gathered}
(D-\overline{\rho}+\varepsilon+\overline{\varepsilon})\mu-(\delta+\overline{\pi}-\overline{\alpha}+\beta)\pi-\sigma\lambda+\nu\kappa- \\
-2\Lambda-\Psi_2=0,
\end{gathered}
\tag{$B^{s+}.8$}
$$

$$
\begin{gathered}
(D+3\varepsilon+\overline{\varepsilon})\nu-(\Delta+\mu+\gamma-\overline{\gamma})\pi-\mu\overline{\tau}-(\overline{\pi}+\tau)\lambda- \\
-\Psi_3-\Phi_{21}=0,
\end{gathered}
\tag{$B^{s+}.9$}
$$

$$(\Delta + \mu + \overline{\mu} + 3\gamma - \overline{\gamma})\lambda - (\overline{\delta} + 3\alpha + \overline{\beta} + \pi - \overline{\tau})\nu + \Psi_4 = 0, \qquad (B^{s+}.10)$$

$$(\delta - \overline{\alpha} - \beta - \tau)\rho - (\delta - 3\alpha + \overline{\beta})\sigma + \tau\overline{\rho} - (\mu - \overline{\mu})\kappa + \\ + \Psi_1 - \Phi_{01} = 0, \qquad (B^{s+}.11)$$

$$(\delta - \overline{\alpha} + 2\beta)\alpha - (\overline{\delta} + \overline{\beta})\beta - \mu\rho + \sigma\lambda - (\rho - \overline{\rho})\gamma - \\ -(\mu - \overline{\mu})\varepsilon - \Lambda + \Psi_2 - \Phi_{11} = 0, \qquad (B^{s+}.12)$$

$$(\delta - \overline{\alpha} + 3\beta)\lambda - (\overline{\delta} + \pi + \alpha + \overline{\beta})\mu - (\rho - \overline{\rho})\nu + \\ + \pi\overline{\mu} + \Psi_3 - \Phi_{21} = 0, \qquad (B^{s+}.13)$$

$$(\delta - \tau + \overline{\alpha} + \beta)\gamma - (\Delta - \gamma + \overline{\gamma} + \mu)\beta - \mu\tau + \sigma\nu + \\ + \varepsilon\overline{\nu} - \alpha\overline{\lambda} - \Phi_{12} = 0, \qquad (B^{s+}.14)$$

$$(\delta - \tau + 3\beta + \overline{\alpha})\nu - (\Delta + \mu + \gamma + \overline{\gamma})\mu - \lambda\overline{\lambda} + \\ + \pi\overline{\nu} - \Phi_{22} = 0, \qquad (B^{s+}.15)$$

$$(\delta - \tau - \beta + \overline{\alpha})\tau - (\Delta + \mu - 3\gamma + \overline{\gamma})\sigma - \overline{\lambda}\rho + \\ + \kappa\overline{\nu} - \Phi_{02} = 0, \qquad (B^{s+}.16)$$

$$(\Delta + \overline{\mu} - \gamma - \overline{\gamma})\rho - (\overline{\delta} + \overline{\beta} - \alpha - \overline{\tau})\tau + \sigma\lambda - \\ -\nu\kappa + 2\Lambda + \Psi_2 = 0, \qquad (B^{s+}.17)$$

$$(\Delta - \overline{\gamma} + \overline{\mu})\alpha - (\overline{\delta} + \overline{\beta} - \overline{\tau})\gamma - (\rho + \varepsilon)\nu + \\ + (\tau + \beta)\lambda + \Psi_3 = 0. \qquad (B^{s+}.18)$$

In addition to these equations, the second structural Cartan equations (B) include the complex conjugate equations

$$R^{+}_{kn} + 2\nabla_{[k}T^{+}_{n]} - [T^{+}_{k}, T^{+}_{n}] = 0. \qquad (B^{s-})$$

We can write these equations in terms of components by replacing the equations $(B^{s+}.1)$–$(B^{s+}.18)$ by their complex conjugate equations.

6 Connection of structural Cartan equations of A_4 geometry with the NP formalism

In 1962 Newman and Penrose [3] put forward a system of nonlinear spinor equations, which appeared to be extremely convenient in the search for novel solutions of Einstein's equations. In the work [10] by the author of this book is was shown that the equations of the Newman-Penrose formalism coincide with the structural Cartan equations of the geometry of absolute parallelism. Indeed, with spinor Carmeli matrices $T_{C\dot{D}}$ one can connect the spintensor $T_{FAC\dot{D}}$ using the relationships

$$(T_{C\dot{D}})_A{}^P = T_A{}^P{}_k \sigma^k{}_{C\dot{D}} = T^P{}_{AC\dot{D}} = -\varepsilon^{PF} T_{FAC\dot{D}}. \qquad (123)$$

Using the matrix elements (162) of the Carmeli matrices and the fundamental spinor

$$\varepsilon^{AB} = \varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we will obtain the following notation for the components of the spintensor $T_{ABC\dot{D}}$:

$$T_{ABC\dot{D}} = \begin{array}{c|cccc} & & C\dot{D} & & \\ \cline{2-5} AB & 0\dot{0} & 0\dot{1} & 1\dot{0} & 1\dot{1} \\ \hline 00 & \kappa & \sigma & \rho & \tau \\ (01) & \varepsilon & \beta & \alpha & \gamma \\ 11 & \pi & \mu & \lambda & \nu \end{array}. \tag{124}$$

Proposition 6.9. First structural Cartan equations of the A_4 geometry coincide with the "coordinate equations" [3]

$$\begin{aligned}\partial_{A\dot{B}}\sigma^i_{C\dot{D}} - \partial_{C\dot{D}}\sigma^i_{A\dot{B}} = \varepsilon^{PQ}(T_{PAC\dot{D}}\sigma^i_{QB} - T_{PCA\dot{B}}\sigma^i_{QD}) + \\ +\varepsilon^{\dot{R}\dot{S}}(\overline{T}_{\dot{R}\dot{B}\dot{D}C}\sigma^i_{A\dot{S}} - \overline{T}_{\dot{R}\dot{D}\dot{B}A}\sigma^i_{C\dot{S}})\end{aligned} \tag{125}$$

in the Newman-Penrose formalism.

Proof. We will write the structural Cartan equations (A) of the geometry of absolute parallelism as

$$\begin{aligned}\partial_{C\dot{D}}\sigma^i_{A\dot{B}} - \partial_{A\dot{B}}\sigma^i_{C\dot{D}} = (T_{C\dot{D}})_A{}^P\sigma^i_{P\dot{B}} + \sigma^i_{A\dot{R}}(T^+_{\dot{D}C})^{\dot{R}}{}_{\dot{B}} - \\ -(T_{A\dot{B}})_C{}^P\sigma^i_{P\dot{D}} - \sigma^i_{C\dot{R}}(T^+_{\dot{B}A})^{\dot{R}}{}_{\dot{D}}.\end{aligned} \tag{126}$$

Using the relationship (123), we will represent the equations (126) as

$$\begin{aligned}\partial_{C\dot{D}}\sigma^i_{A\dot{B}} - \partial_{A\dot{B}}\sigma^i_{C\dot{D}} = -\Big(\varepsilon^{PQ}(T_{PAC\dot{D}}\sigma^i_{QB} - T_{PCA\dot{B}}\sigma^i_{QD}) + \\ +\varepsilon^{\dot{R}\dot{S}}(\overline{T}_{\dot{R}\dot{B}\dot{D}C}\sigma^i_{A\dot{S}} - \overline{T}_{\dot{R}\dot{D}\dot{B}A}\sigma^i_{C\dot{S}})\Big).\end{aligned}$$

It is easily seen that these equations are equivalent to (125).

We now write the well-known decomposition of the Riemannian tensor R_{ijkm} into irreducible representations

$$R_{ijkm} = C_{ijkm} - 2g_{[i[k}R_{m]j]} - \frac{1}{3}Rg_{i[m}g_{k]j}, \tag{127}$$

where C_{ijkm} is the Weyl tensor (10 independent coordinates); R_{ij} is the Ricci tensor (nine independent coordinates); R is the scalar curvature. The spinor representation of these quantities using the Newman-Penrose formalism looks like [11]

$$C_{ijkm} \leftrightarrow \Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}} + \varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}, \tag{128}$$

$$R_{ij} \leftrightarrow 2\Phi_{AB\dot{A}\dot{B}} + 6\varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}, \tag{129}$$

$$R = 24\Lambda, \tag{130}$$

where spinors Ψ_{ABCD} and $\Phi_{AB\dot{A}\dot{B}}$ have the following symmetry properties:

$$\Psi_{ABCD} = \Psi_{(ABCD)}, \quad \Phi_{AB\dot{A}\dot{B}} = \Phi_{(AB)\dot{A}\dot{B}}. \tag{131}$$

By definition the spinors Ψ_{ABCD} and $\Phi_{AB\dot{A}\dot{B}}$ are transformed following the $D(2.0)$ and $D(1.1)$ irreducible representation of the groups $SL^+(2.C)$, respectively.

If we now put in juxtaposition to the Riemann tensor R_{ijkm} a spintensor following the rule

$$R_{ijkm} \leftrightarrow R_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}},$$

then in terms of the spinors (128)-(130) it can be written as

$$\begin{aligned} R_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = \Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}} + \varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} + \\ +\Phi_{AB\dot{C}\dot{D}}\varepsilon_{CD}\varepsilon_{\dot{A}\dot{B}} + \overline{\Phi}_{CD\dot{A}\dot{B}}\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}} + +2\Lambda(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{\dot{A}\dot{D}}\varepsilon_{\dot{C}\dot{D}} + \\ +\varepsilon_{AB}\varepsilon_{CD}\varepsilon_{\dot{A}\dot{D}}\varepsilon_{\dot{B}\dot{C}}). \end{aligned} \tag{132}$$

This spintensor being skew-symmetric in the pair of indices $A\dot{A}$ and $B\dot{B}$, we will write it as

$$R_{A\dot{E}C\dot{B}D\dot{P}F\dot{Q}} = \frac{1}{2}(\varepsilon_{\dot{E}\dot{B}}R_{ACD\dot{P}F\dot{Q}} + \varepsilon_{AC}\overline{R}_{\dot{E}\dot{B}D\dot{P}F\dot{Q}}), \tag{133}$$

where

$$R_{ACD\dot{P}F\dot{Q}} = \frac{1}{2}\varepsilon^{\dot{E}\dot{B}}R_{A\dot{E}C\dot{B}D\dot{P}F\dot{Q}}, \tag{134}$$

$$\overline{R}_{\dot{E}\dot{B}D\dot{P}F\dot{Q}} = \frac{1}{2}\varepsilon^{AC}R_{A\dot{E}C\dot{B}D\dot{P}F\dot{Q}}. \tag{135}$$

Substituting into these relationships the equality (132) gives

$$R_{ACD\dot{P}F\dot{Q}} = \Psi_{ACDF}\varepsilon_{\dot{P}\dot{Q}} + \Phi_{AC\dot{Q}\dot{P}}\varepsilon_{FD} + \Lambda\varepsilon_{\dot{P}\dot{Q}}(\varepsilon_{CD}\varepsilon_{AF} + \varepsilon_{AD}\varepsilon_{CF}), \tag{136}$$

$$\overline{R}_{\dot{E}\dot{B}D\dot{B}PQ} = \varepsilon_{DP}\overline{\Psi}_{\dot{E}\dot{B}\dot{P}\dot{Q}} + \overline{\Phi}_{\dot{B}\dot{E}PD}\varepsilon_{\dot{Q}\dot{P}} + \Lambda\varepsilon_{DP}(\varepsilon_{\dot{B}\dot{P}}\varepsilon_{\dot{E}\dot{Q}} + \varepsilon_{\dot{E}\dot{P}}\varepsilon_{\dot{B}\dot{Q}}). \tag{137}$$

Proposition 6.10. The second structural Cartan equations (B^{s+}) are equivalent to the equations [3][40]

$$\begin{aligned} \Psi_{ACDF}\varepsilon_{\dot{E}\dot{B}} + \Phi_{AC\dot{B}\dot{E}}\varepsilon_{FD} + \Lambda\varepsilon_{\dot{E}\dot{B}}(\varepsilon_{CD}\varepsilon_{AF} + \\ +\varepsilon_{AD}\varepsilon_{CF}) - \partial_{D\dot{B}}T_{ACF\dot{E}} + \partial_{F\dot{E}}T_{ACD\dot{B}} + \\ +\varepsilon^{PQ}(T_{APD\dot{B}}T_{QCF\dot{E}} + T_{ACP\dot{B}}T_{QDF\dot{E}} - T_{APF\dot{E}}T_{QCD\dot{B}} - \\ -T_{ACP\dot{E}}T_{QFD\dot{B}}) + \\ +\varepsilon^{\dot{R}\dot{S}}(T_{ACD\dot{R}}\overline{T}_{\dot{S}\dot{B}\dot{E}F} - T_{ACF\dot{R}}\overline{T}_{\dot{S}\dot{E}\dot{B}D}) = 0 \end{aligned} \tag{138}$$

in the Newman-Penrose formalism.
Proof. We write the equations (B^{s+}) in terms of the Carmeli matrices

$$\begin{aligned} R_{F\dot{E}D\dot{B}} = \partial_{D\dot{B}}T_{F\dot{E}} - \partial_{F\dot{E}}T_{D\dot{B}} - (T_{D\dot{B}})_F{}^S T_{S\dot{B}} - \\ -(T^+_{\dot{E}D})^{\dot{F}}{}_{\dot{B}}T_{F\dot{F}} + (T_{F\dot{E}})_D{}^S T_{S\dot{B}} + \\ +(T^+_{\dot{E}F})^{\dot{F}}{}_{\dot{B}}T_{D\dot{F}} + [T_{F\dot{E}}, T_{D\dot{B}}]. \end{aligned} \tag{139}$$

Using the relationships (123), we can represent the equations (139) as

$$\begin{aligned} R_{ACF\dot{E}D\dot{B}} - \partial_{D\dot{B}}T_{ACE\dot{F}} + \partial_{E\dot{F}}T_{ACD\dot{B}} + T^S{}_{FD\dot{B}}T_{ACS\dot{E}} + \\ +\overline{T}^{\dot{F}}{}_{\dot{E}\dot{B}D}T_{ACF\dot{F}} - T^S{}_{DF\dot{E}}T_{ACS\dot{B}} - \overline{T}^{\dot{F}}{}_{\dot{B}\dot{E}F}T_{ACD\dot{F}} + \\ +\varepsilon^{PQ}(T_{APD\dot{B}}T_{QCF\dot{E}} - T_{APF\dot{E}}T_{QCD\dot{B}}) = 0, \end{aligned}$$

or as

$$
\begin{gathered}
R_{ACF\dot{E}D\dot{B}} - \partial_{D\dot{B}} T_{ACE\dot{F}} + \partial_{E\dot{F}} T_{ACD\dot{B}} + \varepsilon^{PQ}(T_{APD\dot{B}} T_{QCF\dot{E}} + \\
+ T_{ACP\dot{B}} T_{QDF\dot{E}} - T_{APF\dot{E}} T_{QCD\dot{B}} - T_{ACP\dot{E}} T_{QFD\dot{B}}) + \\
+ \varepsilon^{\dot{R}\dot{S}}(T_{ACD\dot{R}} \overline{T}_{\dot{S}\dot{B}\dot{E}F} - T_{ACF\dot{R}} \overline{T}_{\dot{S}\dot{E}\dot{B}D}) = 0,
\end{gathered} \tag{140}
$$

where we have introduced the spinor indices in the matrices $R_{A\dot{B}C\dot{D}}$ and $T_{A\dot{B}}$ following the rule

$$
\begin{gathered}
R_{A\dot{B}C\dot{D}} \to R_{EFA\dot{B}C\dot{D}} = R_{EFkn} \sigma^k_{A\dot{B}} \sigma^n_{C\dot{D}}, \\
T_{A\dot{B}} \to T_{CDA\dot{B}} = T_{CDk} \sigma^k_{A\dot{B}}.
\end{gathered} \tag{141}
$$

Substituting into (140) the relationship (136), we will arrive at the equations (138).

Spintensors Ψ_{ABCE} and $\Phi_{AB\dot{C}\dot{E}}$ have the following notation for their components [38]:

$$
\Psi_{ABCE} = \begin{array}{c|ccc}
 & & CE & \\
AB & 00 & 01 & 11 \\
\hline
00 & \Psi_0 & \Psi_1 & \Psi_2 \\
01 & - & - & \Psi_3 \\
11 & - & - & \Psi_4
\end{array}, \tag{142}
$$

$$
\Phi_{AB\dot{C}\dot{E}} = \begin{array}{c|ccc}
 & & \dot{C}\dot{E} & \\
AB & \dot{0}\dot{0} & \dot{0}\dot{1} & \dot{1}\dot{1} \\
\hline
00 & \Phi_{00} & \Phi_{01} & \Phi_{02} \\
01 & \Phi_{10} & \Phi_{11} & \Phi_{12} \\
11 & \Phi_{20} & \Phi_{21} & \Phi_{22}
\end{array}, \tag{143}
$$

$$
\Lambda = \overline{\Lambda}. \tag{144}
$$

Using the relationships (114), (115), (124), we can expand the equations (126) of the Newman-Penrose formalism component by component to arrive at the equations $(A.1)$ – $(A.8)$ plus the complex conjugate equations. Using the relationships (142)-(144) and (114) we also can expand the equations (138) of the Newman-Penrose formalism componentwise. We will thus end up with the equations $(B^{s+}.1)$–$(B^{s+}.18)$.

The spinor counterpart of the dual Riemann tensor

$$
\overset{*}{R}_{ijkm} = \frac{1}{2} \varepsilon_{km}{}^{sp} R_{ijsp} \tag{145}
$$

can be written as

$$
\begin{aligned}
\overset{*}{R}_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = i \Big(& \varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} - \Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}} - \\
& - \overline{\Phi}_{CD\dot{A}\dot{B}}\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}} + \Phi_{AB\dot{C}\dot{D}}\varepsilon_{CD}\varepsilon_{\dot{A}\dot{B}} + \\
& + 2\Lambda(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{D}\dot{C}} + \varepsilon_{AB}\varepsilon_{CD}\varepsilon_{\dot{A}\dot{C}}\varepsilon_{\dot{B}\dot{D}})\Big).
\end{aligned} \tag{146}
$$

It follows that

$$
\begin{aligned}
\overset{*}{R}_{A\dot{B}C\dot{D}EF} = \frac{1}{2}\varepsilon^{\dot{P}\dot{Q}} R_{A\dot{B}C\dot{D}E\dot{P}F\dot{Q}} = -i\,(& -\varepsilon_{\dot{B}\dot{D}}\Psi_{ACEF} + \\
& + \varepsilon_{EF}\Phi_{AC\dot{B}\dot{D}} + \Lambda\varepsilon_{\dot{B}\dot{D}}(\varepsilon_{AE}\varepsilon_{CF} + \varepsilon_{CE}\varepsilon_{AF})),
\end{aligned} \tag{147}
$$

also

$$\overset{*}{R}_{A\dot{B}C\dot{D}\dot{P}\dot{Q}}=\frac{1}{2}\varepsilon^{EF}R_{A\dot{B}C\dot{D}E\dot{P}F\dot{Q}}=i\left(\varepsilon_{\dot{A}\dot{C}}\overline{\Psi}_{\dot{B}\dot{D}\dot{P}\dot{Q}}-\right.$$
$$\left.-\varepsilon_{\dot{P}\dot{Q}}\overline{\Phi}_{AC\dot{B}\dot{D}}+\Lambda\varepsilon_{AC}(\varepsilon_{\dot{D}\dot{P}}\varepsilon_{\dot{B}\dot{Q}}+\varepsilon_{\dot{B}\dot{P}}\varepsilon_{\dot{D}\dot{Q}})\right). \tag{148}$$

The dual Weyl tensor $\overset{*}{C}_{ijkm}$ corresponds to the spintensor of the form

$$\overset{*}{C}_{ijkm}\leftrightarrow\overset{*}{C}_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}}=i(\varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}-\Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}}).$$

The self-dual spintensor $R_{A\dot{B}C\dot{D}EF}$ will be

$$R_{A\dot{B}C\dot{D}EF}=i\;\overset{*}{R}_{A\dot{B}C\dot{D}EF}=\Psi_{ACEF}\varepsilon_{\dot{B}\dot{D}}, \tag{149}$$

whereas the anti-self-dual tensor is

$$\overline{R}_{A\dot{B}C\dot{D}\dot{P}\dot{Q}}=i\;\overset{*}{R}_{A\dot{B}C\dot{D}\dot{P}\dot{Q}}=\varepsilon_{AC}\overline{\Psi}_{\dot{E}\dot{B}\dot{P}\dot{Q}}. \tag{150}$$

Proposition 6.11. The second Bianchi identities (D^{s+}) of the A_4 geometry in the spinor Δ-basis can be represented as

$$\frac{1}{2i}\varepsilon_{C\dot{D}}{}^{F\dot{E}G\dot{H}R\dot{X}}\partial_{P\dot{X}}R_{ABG\dot{H}F\dot{E}}-\Psi_{ABCR}T^{R}{}_{F}{}^{F}{}_{\dot{D}}-$$
$$-3\Psi_{RPB(A}T_{C)}{}^{RP}{}_{\dot{D}}+\Phi_{RB\dot{D}\dot{X}}T_{A}{}^{R}{}_{C}{}^{\dot{X}}+$$
$$+\Phi_{AB\dot{X}\dot{E}}\overline{T}^{\dot{X}}{}_{\dot{D}}{}^{\dot{E}}{}_{C}+\Phi_{AB\dot{D}\dot{X}}\overline{T}^{\dot{X}}{}_{\dot{E}}{}^{\dot{E}}{}_{C}=0, \tag{151}$$

where

$$\varepsilon^{C\dot{D}F\dot{E}G\dot{H}R\dot{X}}=-i(\varepsilon^{CG}\varepsilon^{RF}\varepsilon^{\dot{D}\dot{E}}\varepsilon^{\dot{H}\dot{X}}-\varepsilon^{CF}\varepsilon^{GR}\varepsilon^{\dot{D}\dot{H}}\varepsilon^{\dot{E}\dot{X}}). \tag{152}$$

Proof. We will write the equations (79) as

$$\nabla^{n}\;\overset{*}{R}_{ACkn}-\overset{*}{R}_{EAkn}\;T_{C}{}^{En}-\overset{*}{R}_{AEkn}\;T_{C}{}^{En}=0. \tag{153}$$

Multiplying these equations by $\sigma^{k}_{C\dot{D}}$ gives

$$\partial^{F\dot{E}}\;\overset{*}{R}_{BAC\dot{D}F\dot{E}}+\overset{*}{R}_{BAC\dot{D}F\dot{E}}\;\nabla^{n}\sigma_{n}{}^{F\dot{E}}+$$
$$+\;\overset{*}{R}_{BAR\dot{S}F\dot{E}}\;\sigma^{k}_{C\dot{D}}\partial^{F\dot{E}}\sigma_{k}{}^{R\dot{S}}-$$
$$-\;\overset{*}{R}_{BPC\dot{D}F\dot{E}}\;T_{A}{}^{PF\dot{E}}-\overset{*}{R}_{PAC\dot{D}F\dot{E}}\;T_{B}{}^{PF\dot{E}}=0. \tag{154}$$

Here we have used the relationships (94) and (133). Substituting into (154) the relationship (148), we will get

$$iD_{ABC\dot{D}}+A_{F\dot{E}P\dot{R}}{}^{i}\left(\sigma_{iC\dot{D}}R_{BA}{}^{P\dot{R}F\dot{E}}-2\sigma_{i}{}^{P\dot{R}}R_{BAC\dot{D}}{}^{F\dot{E}}\right)=0,$$

where $A_{F\dot{E}P\dot{R}}{}^{i}$ stands for the equations (125), rewritten as

$$A_{A\dot{B}C\dot{D}}{}^{i}=\partial_{A\dot{B}}\sigma^{i}{}_{C\dot{D}}-\partial_{C\dot{D}}\sigma^{i}{}_{A\dot{B}}-{}-\varepsilon^{PQ}\left(T_{PAC\dot{D}}\sigma^{i}{}_{Q\dot{B}}-T_{PCA\dot{B}}\sigma^{i}{}_{Q\dot{D}}\right)-$$
$$-\varepsilon^{\dot{R}\dot{S}}\left(\overline{T}_{\dot{R}\dot{B}\dot{D}C}\sigma^{i}{}_{A\dot{S}}-\overline{T}_{\dot{R}\dot{D}\dot{B}A}\sigma^{i}{}_{C\dot{S}}\right)=0, \tag{155}$$

and $D_{ABC\dot{D}} = 0$ defines the equations (151)

$$D_{ABC\dot{D}} = \frac{1}{2i}\varepsilon_{C\dot{D}}{}^{F\dot{E}G\dot{H}R\dot{X}}\partial_{R\dot{X}}R_{ABG\dot{H}F\dot{E}} -$$
$$-\Psi_{ABCR}T^{R}{}_{F}{}^{F}{}_{\dot{D}} - 3\Psi_{RPB(A}T_{C)}{}^{RP}{}_{\dot{D}} + \Phi_{RB\dot{D}\dot{X}}T_{A}{}^{R}{}_{C}{}^{\dot{X}} +$$
$$+\Phi_{AB\dot{X}\dot{E}}\overline{T}^{\dot{X}}{}_{\dot{D}}{}^{\dot{E}}{}_{C} + \Phi_{AB\dot{D}\dot{X}}\overline{T}^{\dot{X}}{}_{\dot{E}}{}^{\dot{E}}{}_{C} = 0,$$

which proves the Proposition.

Proposition 6.12. The second Bianchi identities (151) of the A_4 geometry coincide with the Bianchi identities in the work by Newman-Penrose [3].

$$\partial^{P}{}_{\dot{D}}\Psi_{ABPC} - \partial^{X}{}_{(C}\Phi_{AB)\dot{D}\dot{X}} - 3\Psi_{PR(AB}T_{C)}{}^{PR}{}_{\dot{D}} -$$
$$-\Psi_{ABCP}T^{P}{}_{R}{}^{R}{}_{\dot{D}} + 2T^{P}{}_{(AB}{}^{\dot{X}}\Phi_{C)P\dot{X}\dot{D}} -$$
$$-\overline{T}_{\dot{X}\dot{D}\dot{V}(A}\Phi_{BC)}{}^{\dot{X}\dot{V}} - \overline{T}_{\dot{X}}{}^{\dot{V}}{}_{\dot{V}(A}\Phi_{BC)}{}^{\dot{X}}{}_{\dot{D}} = 0, \qquad (156)$$

$$3\partial_{A\dot{B}}\Lambda + \partial^{P\dot{X}}\Phi_{AP\dot{B}\dot{X}} - \varepsilon^{\dot{V}\dot{W}}\left(\Phi_{AP}{}^{\dot{X}}{}_{\dot{W}}\overline{T}_{\dot{B}\dot{X}\dot{V}}{}^{P} +\right.$$
$$\left.+\Phi_{AP\dot{B}}{}^{\dot{X}}\overline{T}_{\dot{X}\dot{W}\dot{V}}{}^{P}\right) + \Phi_{PR\dot{B}}{}^{\dot{X}}T_{A}{}^{PR}{}_{\dot{X}} +$$
$$+\Phi_{AP\dot{B}}{}^{\dot{X}}T^{P}{}_{R}{}^{R}{}_{\dot{X}} = 0 \qquad (157)$$

Proof. Using (147) and the equality

$$\overset{*}{R}_{ABC\dot{D}}{}^{R\dot{X}} = \frac{1}{2}\varepsilon_{C\dot{D}}{}^{F\dot{E}G\dot{H}R\dot{X}}R_{ABG\dot{H}F\dot{E}},$$

we find that in (151)

$$\frac{1}{2i}\varepsilon_{C\dot{D}}{}^{F\dot{E}G\dot{H}R\dot{X}}\partial_{P\dot{X}}R_{ABG\dot{H}F\dot{E}} = \partial_{P\dot{X}}R_{ABC\dot{D}}{}^{R\dot{X}} =$$
$$= \partial_{P\dot{X}}\left(\varepsilon_{\dot{D}}{}^{\dot{X}}\Psi_{ABC}{}^{R} - \varepsilon_{AB}\Phi_{C}{}^{R}{}_{\dot{D}}{}^{\dot{X}} - \Lambda\varepsilon_{\dot{D}}{}^{\dot{X}}(\varepsilon_{CA}\varepsilon_{B}{}^{R} + \varepsilon_{BA}\varepsilon_{C}{}^{R})\right).$$

Substituting this relationship into (151) gives

$$\partial^{P}{}_{\dot{D}}\Psi_{ABPC} - \partial_{C}{}^{\dot{X}}\Phi_{AB\dot{D}\dot{X}} + 2\varepsilon_{C(A}\partial_{B)\dot{D}}\Lambda - \Psi_{ABCR}T^{R}{}_{F}{}^{F}{}_{\dot{D}} -$$
$$-3\Psi_{RPB(A}T_{C)}{}^{RP}{}_{\dot{D}} + \Phi_{RB\dot{D}\dot{X}}T_{A}{}^{R}{}_{C}{}^{\dot{X}} + \Phi_{AB\dot{X}\dot{E}}\overline{T}^{\dot{X}}{}_{\dot{D}}{}^{\dot{E}}{}_{C} +$$
$$+\Phi_{AB\dot{D}\dot{X}}\overline{T}^{\dot{X}}{}_{\dot{E}}{}^{\dot{E}}{}_{C} = 0. \qquad (158)$$

The part of (158) symmetrical in the indices C and B can be written as (156); and the part skew-symmetrical in these indices looks like (157).

By writing the second Bianchi identities (D^{s+}) of the A_4 geometry component by component, we obtain [3]

$$(D - 4\rho - 2\varepsilon)\Psi_1 - (\overline{\delta} - 4\alpha + \pi)\Psi_0 +$$
$$+3\kappa\Psi_2 + (\delta - 2\beta - 2\overline{\alpha} + \overline{\pi})\Phi_{00} -$$

$$-(D-2\overline{\rho}-2\varepsilon)\Phi_{01}-2\kappa\Phi_{11}+$$
$$+2\sigma\Phi_{10}-\overline{\kappa}\Phi_{02}=0, \qquad (D^{s+}.1)$$

$$(D-3\rho)\Psi_2-(\overline{\delta}+2\pi-2\alpha)\Psi_1+$$
$$+2\kappa\Psi_3+\lambda\Psi_0+(\delta-2\overline{\alpha}+$$
$$+\overline{\pi})\Phi_{10}-(D-2\overline{\rho})\Phi_{11}-\kappa\Phi_{21}-$$
$$-\overline{\kappa}\Phi_{12}-\mu\Phi_{00}+\pi\Phi_{01}+\sigma\Phi_{20}-D\Lambda=0, \qquad (D^{s+}.2)$$

$$(D-2\rho+2\varepsilon)\Psi_3-(\overline{\delta}+3\pi)\Psi_2+$$
$$+2\lambda\Psi_1+\kappa\Psi_4+(\delta-2\overline{\alpha}+2\beta+$$
$$+\overline{\pi})\Phi_{20}-(D-2\overline{\rho}+2\varepsilon)\Phi_{21}-$$
$$-2\mu\Phi_{10}+2\pi\Phi_{11}-\overline{\kappa}\Phi_{22}-2\delta\Lambda=0, \quad (D^{s+}.3)$$

$$(\delta-4\tau-2\beta)\Psi_1-(\Delta-4\gamma+\mu)\Psi_0+$$
$$+3\sigma\Psi_2+(\delta-2\beta+2\overline{\pi})\Phi_{01}-(D-2\varepsilon+2\overline{\varepsilon}-$$
$$-\overline{\rho})\Phi_{02}-2\kappa\Phi_{12}+2\sigma\Phi_{11}-$$
$$-\overline{\lambda}\Phi_{00}=0, \qquad (D^{s+}.4)$$

$$(\delta-3\tau)\Psi_2-(\Delta+2\mu-2\gamma)\Psi_1+2\sigma\Psi_3+$$
$$+\nu\Psi_0+(\delta+2\overline{\pi})\Phi_{11}-(D+2\overline{\varepsilon}-$$
$$-\overline{\rho})\Phi_{12}-\kappa\Phi_{22}-\mu\Phi_{01}+\pi\Phi_{02}+$$
$$+\sigma\Phi_{21}-\overline{\lambda}\Phi_{10}-\delta\Lambda=0, \qquad (D^{s+}.5)$$

$$(\delta+2\beta-2\tau)\Psi_3-(\Delta+3\mu)\Psi_2+2\nu\Psi_1+$$
$$+\sigma\Psi_4+(\delta+2\beta+2\overline{\pi})\Phi_{21}-$$
$$-(D+2\varepsilon+2\overline{\varepsilon}-\overline{\rho})\Phi_{22}-$$
$$-2\mu\Phi_{11}+2\pi\Phi_{12}-\overline{\lambda}\Phi_{20}-2\Delta\Lambda=0, \quad (D^{s+}.6)$$

$$(D+4\varepsilon-\rho)\Psi_4-(\overline{\delta}+4\pi+2\alpha)\Psi_3+$$
$$+3\lambda\Psi_2+(\Delta+2\gamma-2\overline{\gamma}+\overline{\mu})\Phi_{20}-$$
$$-(\overline{\delta}+2\alpha-2\overline{\tau})\Phi_{21}-2\nu\Phi_{10}+$$
$$+2\lambda\Phi_{11}-\overline{\sigma}\Phi_{22}=0, \qquad (D^{s+}.7)$$

$$(\delta+4\beta-\tau)\Psi_4-(\Delta+2\gamma+4\mu)\Psi_3+3\nu\Psi_2+$$
$$+(\Delta+2\gamma+2\overline{\mu})\Phi_{21}-(\overline{\delta}+2\alpha+$$
$$+2\overline{\beta}-\overline{\tau})\Phi_{22}-2\nu\Phi_{11}+2\lambda\Phi_{12}-$$
$$-\overline{\nu}\Phi_{20}=0, \qquad (D^{s+}.8)$$

$$
\begin{aligned}
&(D-2\rho-2\overline{\rho})\Phi_{11}-(\delta-2\overline{\alpha}-2\tau+\\
&+\overline{\pi})\Phi_{10}-(\delta-2\overline{\tau}-2\alpha+\pi)\Phi_{01}+\\
&+(\Delta+2\gamma-2\overline{\gamma}+\mu+\overline{\mu})\Phi_{00}+\\
&+\overline{\kappa}\Phi_{12}+\kappa\Phi_{21}-\overline{\sigma}\Phi_{02}-\\
&-\sigma\Phi_{20}+3D\Lambda=0, \qquad (D^{s+}.9)
\end{aligned}
$$

$$
\begin{aligned}
&(D-2\rho+2\overline{\varepsilon}-\overline{\rho})\Phi_{12}-\\
&-(\delta+2\overline{\pi}-2\tau)\Phi_{11}-(\overline{\delta}+2\overline{\beta}-\\
&-2\alpha-\overline{\tau}+\pi)\Phi_{02}+(\Delta+2\overline{\mu}-2\gamma+\\
&+\mu)\Phi_{01}+\kappa\Phi_{22}-\overline{\nu}\Phi_{00}-\\
&-\overline{\lambda}\Phi_{10}-\sigma\Phi_{21}+3\delta\Lambda=0, \qquad (D^{s+}.10)
\end{aligned}
$$

$$
\begin{aligned}
&(D+2\varepsilon+2\overline{\varepsilon}-\rho-\overline{\rho})\Phi_{22}-\\
&-(\delta+2\overline{\pi}+2\beta-\tau)\Phi_{21}-(\overline{\delta}+\\
&+2\overline{\beta}+2\pi-\overline{\tau})\Phi_{12}+(\Delta+2\mu+\\
&+2\overline{\mu})\Phi_{11}-\overline{\nu}\Phi_{10}-\nu\Phi_{01}+\\
&+\overline{\lambda}\Phi_{20}+\lambda\Phi_{02}+3\Delta\Lambda=0. \qquad (D^{s+}.11)
\end{aligned}
$$

To arrive at the complete set of the second Bianchi (D) identities of the A_4 geometry, we will have to add to these equations their complex conjugate (D^{s-}).

7 Variational principle of derivation of the structural Cartan equations and the second Bianchi identities of $\mathbf{A_4}$ geometry

To begin with, we will consider the derivation of the structural equations (B) and of the second Bianchi identities (D) for self-dual and anti-self-dual fields of Riemannian curvature, whose Carmeli matrices obey the conditions

$$R_{kn}=\pm i \overset{*}{R}_{kn},$$

$$R^{+}_{kn}=\pm i \overset{*}{R}_{kn},$$

where

$$R_{kn}+2\nabla_{[k}T_{n]}-[T_k,T_n]=0,$$
$$R^{+}_{kn}+2\nabla_{[k}T^{+}_{n]}-[T^{+}_k,T^{+}_n]=0,$$

and

$$\overset{*}{R}_{kn}=\frac{1}{2}\varepsilon^{knps}R_{ps},$$

$$\overset{*}{R}{}^{+}_{kn}=\frac{1}{2}\varepsilon^{knps}R^{+}_{ps}.$$

Let us take the Lagrange function in the form

$$L_1 = -\frac{1}{4}(-g)^{1/2}Tr(R_{kn}R^{kn}) + \text{complex conjugate part.} \tag{159}$$

Varying this expression in T_k and T^+_k, we will arrive at the equations (D)

$$\nabla^n \overset{*}{R}_{kn} + [\overset{*}{R}_{kn}, T^n] = 0, \tag{160}$$

$$\nabla^n \overset{*}{R}{}^{+}_{kn} + [\overset{*}{R}{}^{+}_{kn}, T^{+n}] = 0. \tag{161}$$

For arbitrary fields of Riemannian curvature the Lagrange function looks like

$$L_2 = -\frac{1}{2}(-g)^{1/2}Tr\left(\overset{*}{R}{}^{kn}(-\frac{1}{2}R_{kn} - 2\nabla_{[k}T_{n]} + [T_k, T_n])\right) + \text{c.c. part.} \tag{162}$$

Variation of this Lagrangian in $\overset{*}{R}_{kn}$ and $\overset{*}{R^+}_{kn}$ yields the second Bianchi identities (D)

$$\nabla^n \overset{*}{R}_{kn} + [\overset{*}{R}_{kn}, T^n] = 0, \tag{D^{s+}}$$

$$\nabla^n \overset{*}{R^+}_{kn} + [\overset{*}{R}{}^{+}_{kn}, T^{+n}] = 0. \tag{D^{s-}}$$

On the other hand, variation of the Lagrangian (162) in T_k and $T^+{}_k$ gives the second structural Cartan equations (B) of the A_4 geometry

$$R_{kn} + 2\nabla_{[k}T_{n]} - [T_k, T_n] = 0, \tag{B^{s+}}$$

$$R^+_{kn} + 2\nabla_{[k}T^+_{n]} - [T^+_k, T^+_n] = 0, \tag{B^{s-}}$$

and

$$\overset{*}{R}_{kn} = \frac{1}{2}\varepsilon^{knps}R_{ps},$$

$$\overset{*}{R}{}^{+}_{kn} = \frac{1}{2}\varepsilon^{knps}R^+_{ps}.$$

Independent variables in the Lagrangian (162) are the quantities R_{kn}, R^+_{kn}, T_k, and T^+_k. To obtain from them using the variational principle, the first structural Cartan equations (A) of the A_4 geometry

$$\nabla_{[k}\sigma^{i]} - T_{[k}\sigma^{i]} - \sigma^{[i}T^+_{k]} = 0, \tag{A^s}$$

we will have to introduce into the Lagrangian (162) as independent variables the matrices σ^i. This can be done by modifying the Lagrangian (162) as it has been done in [12].

We now write the equations $(A), (B)$ and (D) in spinor form :

$$(A) \qquad A^i{}_{A\dot{B}C\dot{D}} = 0, \tag{163}$$

$$(B) \quad B_{F\dot{E}ACD\dot{B}} = 0 + \text{c.c. equations}, \tag{164}$$

$$(D) \quad D_{ABC\dot{D}} = 0 + \text{c.c. equations}, \tag{165}$$

where

$$A^i_{A\dot{B}C\dot{D}} = \partial_{A\dot{B}}\sigma^i_{C\dot{D}} - \partial_{C\dot{D}}\sigma^i_{A\dot{B}} - \varepsilon^{PQ}(T_{PAC\dot{D}}\sigma^i_{Q\dot{B}} - \\ -T_{PCA\dot{B}}\sigma^i_{Q\dot{D}}) - \varepsilon^{\dot{R}\dot{S}}(\overline{T}_{\dot{R}\dot{B}\dot{D}C}\sigma^i_{A\dot{S}} - \\ -\overline{T}_{\dot{R}\dot{D}\dot{B}A}\sigma^i_{C\dot{S}}) = 0, \quad (166)$$

$$B_{ACF\dot{E}D\dot{B}} = R_{ACF\dot{E}D\dot{B}} - \partial_{D\dot{B}}T_{ACE\dot{F}} + \partial_{E\dot{F}}T_{ACD\dot{B}} + +\varepsilon^{PQ}(T_{APD\dot{B}}T_{QCF\dot{E}} + \\ +T_{ACP\dot{B}}T_{QDF\dot{E}} - T_{APF\dot{E}}T_{QCD\dot{B}} - T_{ACP\dot{E}}T_{QFD\dot{B}}) + \\ +\varepsilon^{\dot{R}\dot{S}}(T_{ACD\dot{R}}\overline{T}_{\dot{S}\dot{B}\dot{E}F} - T_{ACF\dot{R}}\overline{T}_{\dot{S}\dot{E}\dot{B}D}) = 0, \quad (167)$$

$$D_{ABC\dot{D}} = \frac{1}{2i}\varepsilon_{C\dot{D}}{}^{F\dot{E}G\dot{H}R\dot{X}}\partial_{R\dot{X}}R_{ABG\dot{H}F\dot{E}} - -\Psi_{ABCR}T^R{}_F{}^F{}_{\dot{D}} - 3\Psi_{RPB(A}T_{C)}{}^{RP}{}_{\dot{D}} + \\ +\Phi_{RB\dot{D}\dot{X}}T_A{}^R{}_C{}^{\dot{X}} + \Phi_{AB\dot{X}\dot{E}}\overline{T}^{\dot{X}}{}_{\dot{D}}{}^{\dot{E}}{}_C + \Phi_{AB\dot{D}\dot{X}}\overline{T}^{\dot{X}}{}_{\dot{E}}{}^{\dot{E}}{}_C = 0 \quad (168)$$

and consider the Lagrangian

$$L_3 = \overset{*}{R}{}^B{}_{\dot{Q}}{}^{A\dot{Q}kn}\left((2\nabla_n T_{ABk} + 2T_{PAn}T_B{}^P{}_k) - \frac{1}{4}R_{BPA}{}^P{}_{nk}\right) + \text{c.c. part.} \quad (169)$$

Here $\overset{*}{R}{}^B{}_{\dot{Q}}{}^{A\dot{Q}kn} = \varepsilon^{nkjm}R^B{}_{\dot{Q}}{}^{A\dot{Q}}{}_{jm}$ and ε^{nkjm} is a completely skew-symmetrical Levi-Chivita symbol.

If we take $R^B{}_{\dot{Q}}{}^{A\dot{Q}kn}$ and T_{PAn} to be independent variables and use the conventional variational procedure, we will obtain the following equations:

$$(B^{s+}) \quad \frac{1}{2}R_{B\dot{P}A}{}^{\dot{P}}{}_{kn} - 2\nabla_{[k}T_{|AB|n]} + 2T_{PA[k}T^P{}_{|B|n]} = 0, \quad (170)$$

$$(B^{s-}) \quad \text{complex conjugate equations}, \quad (171)$$

$$(D^{s+}) \quad \nabla^k \overset{*}{R}_{B\dot{Q}A}{}^{\dot{Q}}{}_{nk} - 2\overset{*}{R}_{P\dot{Q}(A}{}^{\dot{Q}}{}_{|nk|}T^{Pk}_{B)} = 0, \quad (172)$$

$$(D^{s-}) \quad \text{complex conjugate equations}. \quad (173)$$

Multiplying equations (170) by $\sigma_{C\dot{D}}{}^n\sigma_{F\dot{E}}{}^n$ gives

$$\partial_{F\dot{E}}T_{ABC\dot{D}} - \partial_{C\dot{D}}T_{ABF\dot{E}} + T_{PAF\dot{E}}T^P_{BA\dot{D}} - T_{PAC\dot{D}}T^P_{BF\dot{E}} - \\ -\frac{1}{2}R_{B\dot{Q}A}{}^P{}_{F\dot{E}C\dot{D}} + T_{ABn}(\partial_{C\dot{D}}\sigma_{F\dot{E}}{}^n - \partial_{F\dot{E}}\sigma_{C\dot{D}}{}^n) = 0. \quad (174)$$

Using the notation (166) and (167), we will write (174) as

$$B_{ACF\dot{E}D\dot{B}} + A^n{}_{CDF\dot{E}}T_{ABn} = 0. \quad (175)$$

We will now multiply (172) by $\sigma_{C\dot{D}}{}^k$ to get the relationship

$$\partial^{F\dot{E}}\overset{*}{R}{}_{B\dot{Q}A}{}^{\dot{Q}}{}_{C\dot{D}E\dot{F}} + \overset{*}{R}_{B\dot{Q}A}{}^{\dot{Q}}{}_{C\dot{D}E\dot{F}}\nabla^k\sigma_k{}^{F\dot{E}} + \\ + \overset{*}{R}_{B\dot{Q}A}{}^{\dot{Q}}{}_{R\dot{S}E\dot{F}}\sigma^n{}_{C\dot{D}}\partial^{F\dot{E}}\sigma^{R\dot{S}}{}_n - \\ - \overset{*}{R}_{B\dot{Q}P}{}^{\dot{Q}}{}_{C\dot{D}E\dot{F}}T_A{}^{PF\dot{E}} - \overset{*}{R}_{P\dot{Q}A}{}^{\dot{Q}}{}_{C\dot{D}E\dot{F}}T_B{}^{PF\dot{E}} = 0 \quad (176)$$

or, from (166) and (167),

$$iD_{AB C\dot{D}} + A_{F\dot{E}P\dot{R}}{}^{n}\left(\frac{1}{2}\sigma_{nC\dot{D}}\overset{*}{R}_{B\dot{Q}A}{}^{\dot{Q}P\dot{R}E\dot{F}} - \sigma_{n}{}^{P\dot{R}}\overset{*}{R}_{B\dot{Q}A}{}^{\dot{Q}}{}_{C\dot{D}}{}^{E\dot{F}}\right) = 0. \tag{177}$$

Here we have also used the relationship

$$\overset{*}{R}_{B\dot{Q}A}{}^{\dot{Q}}{}_{C\dot{D}E\dot{F}} = i\left(2\Psi_{BACF}\varepsilon_{\dot{D}\dot{E}} - 2\varepsilon_{CF}\Phi_{AB\dot{D}\dot{E}} + 2\Lambda\varepsilon_{\dot{D}\dot{E}}(\varepsilon_{BF}\varepsilon_{AC}+ \right.$$
$$\left. +\varepsilon_{BC}\varepsilon_{AF})\right).$$

It is clear that from the Lagrangian (169) it is impossible to obtain the first structural Cartan equations (A) of the A_4 geometry, since it does not contain $\sigma_{C\dot{D}}{}^{n}$.

Let us add to the Lagrangian (169) the term

$$\lambda_{j}^{A\dot{B}C\dot{D}}A^{j}_{A\dot{B}C\dot{D}} \tag{178}$$

where the quantities $\lambda^{A\dot{B}C\dot{D}}$ play the role of Lagrange factors

$$L_4 = L_3 + \lambda_{j}^{A\dot{B}C\dot{D}}A^{j}_{A\dot{B}C\dot{D}} + \text{c.c.}\quad\text{part.} \tag{179}$$

The quantities $\lambda_{j}^{A\dot{B}C\dot{D}}$, just like $A^{j}_{A\dot{B}C\dot{D}}$, are Hermitian matrices, which are skew-symmetrical in the pair of indices [12]$A\dot{B}$ and $C\dot{D}$. Varying the Lagrange density (179) in $\sigma_{C\dot{D}}{}^{n}$ gives [12]

$$A^{j}_{A\dot{B}C\dot{D}} = 0 \tag{180}$$

and

$$D_{ABC\dot{D}} = \sigma_{k}^{P\dot{R}}\sigma_{nB}^{\dot{X}}(\lambda^{n}_{A\dot{X}P\dot{R}} - \overline{\lambda}^{n}_{A\dot{X}P\dot{R}})\sigma^{k}_{C\dot{D}} = 0. \tag{181}$$

Since $\lambda^{n}_{A\dot{X}P\dot{R}}$ are Hermitian matrices, from (181) we have the equations (D^{s+})

$$D_{ABC\dot{D}} = 0. \tag{182}$$

Hence varying the complex conjugate part of the Lagrangian (179) gives

$$\overline{D}_{\dot{A}\dot{B}\dot{C}D} = 0. \tag{183}$$

and of the Lagrangian (179) in $R^{B}{}_{\dot{Q}A}{}^{\dot{Q}kn}$ gives

$$B_{ACF\dot{E}D\dot{B}} + A^{n}{}_{CDF\dot{E}}T_{ABn} = 0 \tag{184}$$

or, from (180),

$$B_{ACF\dot{E}D\dot{B}} = 0. \tag{185}$$

Variation of the complex conjugate part in $\overline{R}^{\dot{B}}{}_{Q}{}^{\dot{A}Qkn}$ yields

$$\overline{B}_{ACF\dot{E}D\dot{B}} = 0. \tag{186}$$

It has thus been shown that from the Lagrangian (179) follow the first and second of the structural Cartan equations of the A_4 geometry (equations (180), (185) and (186)), and also the second of the Bianchi identities (equations (182) and (183)).

8 Decomposition of spinor fields of $\mathbf{A_4}$ geometry into irreducible parts

The Ricci torsion tensor $\Omega_{jk}^{..i}$ of the A_4 space has 24 independent components, and it can be represented as the sum of three irreducible parts as follows:

$$\Omega^{i}_{.jk} = \frac{2}{3}\delta^{i}_{[k}\Omega_{j]} + \frac{1}{3}\varepsilon^{n}_{jks}\hat{\Omega}^{s} + \overline{\Omega}^{i}_{.jk}, \tag{187}$$

where

$$\Omega^{i}_{.jk} = g^{im}g_{ks}\Omega_{mj}^{..s}, \tag{188}$$

and the vector Ω_j, pseudovector $\hat{\Omega}_j$ and the traceless part of the torsion $\overline{\Omega}^{i}_{.jk}$ are given by

$$\Omega_j = \Omega^{i}_{.ji}, \tag{189}$$

$$\hat{\Omega}_j = \frac{1}{2}\varepsilon_{jins}\Omega^{ins}, \tag{190}$$

$$\overline{\Omega}^{s}_{.js} = 0, \quad \overline{\Omega}_{ijs} + \overline{\Omega}_{jsi} + \overline{\Omega}_{sij} = 0. \tag{191}$$

In the spinor basis the spinor representation of the Ricci rotation coefficients $T_{ABC\dot{C}}$ has the form [40]

$$T_{ABC\dot{C}} = \frac{1}{2}\left(A_{ABC\dot{C}} + \frac{1}{3}(\varepsilon_{AC}\alpha_{B\dot{C}} + \varepsilon_{BC}\alpha_{A\dot{C}})\right), \tag{192}$$

where the spinor $A_{ABC\dot{C}}$ is completely symmetrical in the unprimed indices

$$A_{ABC\dot{C}} = A_{(ABC)\dot{C}}, \tag{193}$$

and the spinor $\alpha_{B\dot{C}}$ is given by

$$\alpha_{A\dot{C}} = A_{AB}{}^{B}{}_{\dot{C}}. \tag{194}$$

In turn, the spinor $\alpha_{A\dot{C}}$ can be decomposed into the Hermitian and anti-Hermitian parts:

$$\alpha_{A\dot{C}} = \kappa_{A\dot{C}} - i\mu_{A\dot{C}}, \tag{195}$$

where

$$\kappa_{A\dot{C}} = \frac{1}{2}(\alpha_{A\dot{C}} + \overline{\alpha}_{\dot{A}C}), \quad \mu_{A\dot{C}} = \frac{1}{2}i(\alpha_{A\dot{C}} - \overline{\alpha}_{\dot{A}C}) \tag{196}$$

and

$$\overline{\kappa_{A\dot{C}}} = \overline{\kappa}_{\dot{A}C} = \kappa_{C\dot{A}}, \quad \overline{\mu_{A\dot{C}}} = \overline{\mu}_{\dot{A}C} = \mu_{C\dot{A}}. \tag{197}$$

The irreducible parts of torsion (189)-(191) and the spinors (193)-(197) are related by

$$\Omega_j \longleftrightarrow \kappa_{A\dot{C}}, \tag{198}$$

$$\hat{\Omega}_j \longleftrightarrow \mu_{A\dot{C}}, \tag{199}$$

$$\overline{\Omega}^{k}_{.js} \longleftrightarrow A_{ABC\dot{C}}. \tag{200}$$

Since

$$\Omega_{ijk} = g_{sk}\Omega_{ij}^{..s}, \tag{201}$$

we have

$$\Omega_{A\dot{A}B\dot{B}C\dot{C}} \longleftrightarrow \Omega_{ijk}, \tag{202}$$

$$\Omega_{A\dot{A}B\dot{B}C\dot{C}} = \frac{1}{2}(\Omega_{ABC\dot{C}}\varepsilon_{\dot{A}\dot{B}} + \overline{\Omega}_{\dot{A}\dot{B}\dot{C}C}\varepsilon_{AB}), \tag{203}$$

$$\Omega_{ABC\dot{C}} = A_{C(AB)\dot{C}} + \overline{\alpha}_{\dot{C}(A}\varepsilon_{B)C}. \tag{204}$$

By definition, the spinor $A_{ABC\dot{C}}$ is transformed in the $D(3/2.1/2)$ irreducible representation of the group $SL(2,C)$. Consequently, the spinors $\kappa_{A\dot{C}}$ and $\mu_{A\dot{C}}$ are transformed in the $D(1/2.1/2)$ irreducible representation of the group $SL(2.C)$. Using the relationship (124), we can find the components of the spinors $\kappa_{A\dot{C}}$ and $\mu_{A\dot{C}}$ [13]

$$\kappa_{A\dot{C}} = \begin{pmatrix} \frac{1}{2}(\rho+\overline{\rho}) - \frac{1}{2}(\varepsilon+\overline{\varepsilon}) & \frac{1}{2}(\tau+\beta) + \frac{1}{2}(\overline{\alpha}-\overline{\pi}) \\ \frac{1}{2}(\overline{\tau}-\overline{\beta}) + \frac{1}{2}(\alpha-\pi) & \frac{1}{2}(\gamma+\overline{\gamma}) - \frac{1}{2}(\mu+\overline{\mu}) \end{pmatrix}, \tag{205}$$

$$\mu_{A\dot{C}} = i\begin{pmatrix} \frac{1}{2}(\rho-\overline{\rho}) - \frac{1}{2}(\varepsilon-\overline{\varepsilon}) & \frac{1}{2}(\tau-\beta) - \frac{1}{2}(\overline{\alpha}-\overline{\pi}) \\ -\frac{1}{2}(\overline{\tau}-\overline{\beta}) + \frac{1}{2}(\alpha-\pi) & \frac{1}{2}(\gamma-\overline{\gamma}) - \frac{1}{2}(\mu-\overline{\mu}) \end{pmatrix}. \tag{206}$$

The Riemann tensor represented in terms of irreducible parts is

$$R_{ijkm} = C_{ijkm} + g_{i[k}R_{m]j} + g_{j[k}R_{m]i} + \frac{1}{3}Rg_{i[m}g_{k]j}. \tag{207}$$

In the spinor basis this becomes [3]

$$R_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = \Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}} + +\varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} + \Phi_{AB\dot{C}\dot{D}}\varepsilon_{CD}\varepsilon_{\dot{A}\dot{B}} + \\ +\overline{\Phi}_{CD\dot{A}\dot{B}}\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}} + 2\Lambda(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}} + +\varepsilon_{AB}\varepsilon_{CD}\varepsilon_{\dot{A}\dot{D}}\varepsilon_{\dot{B}\dot{C}}).$$

We also have the following connection:

$$\begin{aligned} C_{ijkm} &\longleftrightarrow \Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}} + \varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}, \\ R_{ij} &\longleftrightarrow 2\Phi_{AB\dot{C}\dot{D}} + 6\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}}, \\ R &\longleftrightarrow 24\Lambda, \end{aligned} \tag{208}$$

where the spinors Ψ_{ABCD}, $\Phi_{AB\dot{C}\dot{D}}$ and Λ meet the following symmetry conditions:

$$\Psi_{ABCD} = \Psi_{(ABCD)}, \quad \Phi_{AB\dot{C}\dot{D}} = \Phi_{(AB)(\dot{C}\dot{D})}, \quad \Lambda = \overline{\Lambda}$$

and belong to the $D(2.0)$, $D(1.1)$ and $D(0.0)$ irreducible representations of the group $SL(2.C)$, respectively.

9 Spinor set of Einstein-Yang-Mills equations

In the first part of the book it was shown that the structural Cartan equations of the geometry of absolute parallelism (A) and (B) can be represented as an extended set of Einstein-Yang-Mills equations

$$\boxed{\begin{array}{lr} \nabla_{[k}e^{a}{}_{j]} + T^{i}_{[kj]}e^{a}{}_{i} = 0, & (A) \\ R_{jm} - \frac{1}{2}g_{jm}R = \nu T_{jm}, & (B.1) \\ C^{i}{}_{jkm} + 2\nabla_{[k}T^{i}{}_{|j|m]} + 2T^{i}_{s[k}T^{s}{}_{|j|m]} = -\nu J^{i}{}_{jkm}. & (B.2) \end{array}} \tag{209}$$

We will write this set of equations in the spinor basis. To this end, we will make use of the Carmeli matrices and the Newman-Penrose spinor formalism. Suppose now we have the right spin A_4 geometry, then its equations (A) and (B) have the form

$$\partial_{C\dot{D}}\sigma^i_{A\dot{B}} - \partial_{A\dot{B}}\sigma^i_{C\dot{D}} = (T_{C\dot{D}})^P_A\sigma^i_{P\dot{B}} +$$
$$+\sigma^i_{A\dot{R}}(T^+_{\dot{D}C})^{\dot{R}}_{\dot{B}} - (T_{A\dot{B}})^P_C\sigma^i_{P\dot{D}} - \sigma^i_{C\dot{R}}(T^+_{\dot{B}A})^{\dot{R}}_{\dot{D}}, \qquad (A^{s+})$$

$$R_{A\dot{B}C\dot{D}} = \partial_{C\dot{D}}T_{A\dot{B}} - \partial_{A\dot{B}}T_{C\dot{D}} - (T_{C\dot{D}})^F_A T_{F\dot{B}} -$$
$$-(T^+_{\dot{D}C})^{\dot{F}}_{\dot{B}}T_{A\dot{F}} + (T_{A\dot{B}})^F_C T_{F\dot{D}} + (T^+_{\dot{B}A})^{\dot{F}}_{\dot{D}}T_{C\dot{F}} + [T_{A\dot{B}}, T_{C\dot{D}}], \qquad (B^{s+})$$

where the components of the matrices $\sigma^i_{A\dot{B}}$, $T_{A\dot{B}}$ and $R_{A\dot{B}C\dot{D}}$ are given by (115), (88) and (103), respectively.

Proposition 6.13. Equations $(B.1)$ in the spinor basis are

$$2\Phi_{AB\dot{C}\dot{D}} + \Lambda\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}} = \nu T_{A\dot{C}B\dot{D}}. \tag{210}$$

Proof. In terms of the irreducible spinors (208) $P-Q$ the components of the spinor matrices $R_{A\dot{B}C\dot{D}}$ are given by [14]

$$(R_{A\dot{B}C\dot{D}})_P{}^Q = \varepsilon_{\dot{D}\dot{B}}\left(\Psi_{CAP}{}^Q - \Lambda(\varepsilon_{PC}\delta^Q_A + \varepsilon_{PA}\delta^Q_C)\right) + \varepsilon_{CA}\Phi_P{}^Q{}_{\dot{D}\dot{B}}, \tag{211}$$

where

$$(C_{A\dot{B}C\dot{D}})_P{}^Q = \varepsilon_{\dot{D}\dot{B}}\Psi_{CAP}{}^Q \tag{212}$$

are the $P-Q$ components of the spinor matrices of the Weyl tensor with the the components

$$C_{0\dot{1}0\dot{0}} = \begin{pmatrix} \Psi_1 & -\Psi_0 \\ \Psi_2 & -\Psi_1 \end{pmatrix}, \quad C_{1\dot{1}1\dot{0}} = \begin{pmatrix} \Psi_3 & -\Psi_2 \\ \Psi_4 & -\Psi_3 \end{pmatrix},$$
$$C_{1\dot{1}0\dot{0}} = \begin{pmatrix} \Psi_2 & -\Psi_1 \\ \Psi_3 & -\Psi_2 \end{pmatrix}, \quad C_{1\dot{0}0\dot{1}} = \begin{pmatrix} -\Psi_2 & \Psi_1 \\ -\Psi_3 & \Psi_2 \end{pmatrix}, \tag{213}$$

and related with the spinor $\Lambda\varepsilon_{\dot{D}\dot{B}}(\varepsilon_{PC}\delta^A{}_Q + \varepsilon_{PA}\delta_C{}^Q)$ and $\varepsilon_{CA}\Phi_P{}^Q{}_{\dot{D}\dot{B}}$ are the trace and traceless parts of the Ricci tensor

$$\Lambda\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}} = -\frac{1}{4}\sigma^k{}_{A\dot{C}}\sigma^n{}_{B\dot{D}}Rg_{kn}, \tag{214}$$

$$\Phi_{AB\dot{C}\dot{D}} = \frac{1}{2}\sigma^k{}_{A\dot{C}}\sigma^n{}_{B\dot{D}}\left(R_{kn} - \frac{1}{4}g_{kn}R\right). \tag{215}$$

Substituting relationships (214) and (215) into (210) and multiplying the resultant expression by $\sigma^{A\dot{C}}{}_k\sigma^{B\dot{D}}{}_n$, we arrive at the equations $(B.1)$.

We now represent the matrix $R_{A\dot{B}C\dot{D}}$ as the sum

$$R_{A\dot{B}C\dot{D}} = C_{A\dot{B}C\dot{D}} + \nu J_{A\dot{B}C\dot{D}}, \tag{216}$$

where the matrix current $J_{A\dot{B}C\dot{D}}$ has the components [15]:

$$J_{0\dot{1}0\dot{0}} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ \frac{1}{6}T & 0 \end{pmatrix}, \quad J_{1\dot{1}1\dot{0}} = \frac{1}{2}\begin{pmatrix} 0 & -\frac{1}{6}T \\ 0 & 0 \end{pmatrix},$$

$$J_{1\dot{0}0\dot{0}} = \frac{1}{2}\left(\begin{array}{cc} T_{1\dot{0}0\dot{0}} & -T_{0\dot{0}0\dot{0}} \\ T_{1\dot{0}1\dot{0}} & -T_{1\dot{0}0\dot{0}} \end{array}\right),$$

$$J_{1\dot{1}0\dot{1}} = \frac{1}{2}\left(\begin{array}{cc} T_{0\dot{1}1\dot{1}} & -T_{0\dot{1}0\dot{1}} \\ T_{1\dot{1}1\dot{1}} & -T_{0\dot{1}1\dot{1}} \end{array}\right), \quad (217)$$

$$J_{1\dot{1}0\dot{0}} = \frac{1}{2}\left(\begin{array}{cc} T_{1\dot{1}0\dot{0}} & -T_{0\dot{1}0\dot{0}} \\ T_{1\dot{0}1\dot{1}} & -T_{1\dot{1}0\dot{0}} \end{array}\right) - \frac{1}{2}\left(\begin{array}{cc} \frac{1}{6}T & 0 \\ 0 & -\frac{1}{6}T \end{array}\right),$$

$$J_{1\dot{0}0\dot{1}} = \frac{1}{2}\left(\begin{array}{cc} T_{1\dot{1}0\dot{0}} & -T_{0\dot{1}0\dot{0}} \\ T_{1\dot{0}1\dot{1}} & -T_{1\dot{1}0\dot{0}} \end{array}\right) + \frac{1}{2}\left(\begin{array}{cc} -\frac{1}{6}T & 0 \\ 0 & \frac{1}{6}T \end{array}\right).$$

Here

$$T_{A\dot{B}C\dot{D}} = \sigma^{k}{}_{A\dot{C}}\sigma^{n}{}_{B\dot{D}}T_{kn}, \quad (218)$$

$$T = g^{jm}T_{jm}, \quad (219)$$

and the energy-momentum tensor T_{kn} is given in terms of the Ricci rotation coefficients by

$$T_{jm} = -\frac{2}{\nu}\left\{\nabla_{[i}T^{i}_{|j|m]} + T^{i}_{s[i}T^{s}_{|j|m]} - \right.$$
$$\left. -\frac{1}{2}g^{pn}g_{jm}\left(\nabla_{[i|}T^{i}_{|p|n]} + T^{i}_{s[i}T^{s}_{|p|n]}\right)\right\}. \quad (220)$$

In the special case where the field $T^{i}{}_{jk}$ is skew-symmetric in all the three indices, the tensor (218) is

$$T_{jm} = \frac{1}{\nu}\left(\hat{\Omega}_j\hat{\Omega}_m - \frac{1}{2}g_{jm}\hat{\Omega}^i\hat{\Omega}_i\right). \quad (221)$$

Multiplying this by $\sigma^{j}{}_{A\dot{C}}\sigma^{m}{}_{B\dot{D}}$ and using (199), we get

$$T_{A\dot{B}C\dot{D}} = \frac{1}{\nu}\left(\mu_{A\dot{B}}\mu_{C\dot{D}} - \frac{1}{2}\varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}}\mu_{P\dot{Q}}\mu^{P\dot{Q}}\right). \quad (222)$$

In addition, we obtain

$$T = g^{jm}T_{jm} = -\frac{1}{\nu}\hat{\Omega}_j\hat{\Omega}^j = -\frac{1}{\nu}\mu_{P\dot{Q}}\mu^{P\dot{Q}}. \quad (223)$$

Hence the "density of spinor matter" is

$$\rho = -\frac{1}{\nu c^2}\mu_{P\dot{Q}}\mu^{P\dot{Q}}. \quad (224)$$

We substitute (6.217) into the spinor equations (B^{s+}) go get

$$2\Phi_{AB\dot{C}\dot{D}} + \Lambda\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}} = \nu T_{A\dot{C}B\dot{D}}, \quad (B^{s+}.1)$$

$$C_{A\dot{B}C\dot{D}} - \partial_{C\dot{D}}T_{A\dot{B}} + \partial_{A\dot{B}}T_{C\dot{D}} + (T_{C\dot{D}})^{F}_{A}T_{F\dot{B}} + (T^{+}_{\dot{D}C})^{\dot{F}}_{\dot{B}}T_{A\dot{F}} -$$
$$-(T_{A\dot{B}})^{F}_{C}T_{F\dot{D}} - (T^{+}_{\dot{B}A})^{\dot{F}}_{\dot{D}}T_{C\dot{F}} - [T_{A\dot{B}}, T_{C\dot{D}}] = -\nu J_{A\dot{B}C\dot{D}}. \quad (B^{s+}.2)$$

To conclude, we will write the extended set of Einstein-Yang-Mills equations as

$$\boxed{\begin{array}{lr}
\partial_{C\dot{D}}\sigma^i_{\ A\dot{B}} - \partial_{A\dot{B}}\sigma^i_{\ C\dot{D}} = (T_{C\dot{D}})^P_A\sigma^i_{\ P\dot{B}} + \sigma^i_{\ A\dot{R}}(T^+_{\dot{D}C})^{\dot{R}}_{\dot{B}} - & \\
-(T_{A\dot{B}})^P_C\sigma^i_{\ P\dot{D}} - \sigma^i_{\ C\dot{R}}(T^+_{\dot{B}A})^{\dot{R}}_{\dot{D}}, & (A^s) \\
2\Phi_{AB\dot{C}\dot{D}} + \Lambda\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}} = \nu T_{A\dot{C}B\dot{D}}, & (B^{s+}.1) \\
C_{A\dot{B}C\dot{D}} - \partial_{C\dot{D}}T_{A\dot{B}} + \partial_{A\dot{B}}T_{C\dot{D}} + (T_{C\dot{D}})^F_A T_{F\dot{B}} + (T^+_{\dot{D}C})^{\dot{F}}_{\dot{B}}T_{A\dot{F}} - & \\
-(T_{A\dot{B}})^F_C T_{F\dot{D}} - (T^+_{\dot{B}A})^{\dot{F}}_{\dot{D}}T_{C\dot{F}} - [T_{A\dot{B}}, T_{C\dot{D}}] = -\nu J_{A\dot{B}C\dot{D}}. & (B^{s+}.2)
\end{array}}$$

where the spinor indices take on the values $A, B, D\ldots = 0, 1,\ \dot{A}, \dot{B}, \dot{D}\ldots = \dot{0}, \dot{1}$.

10 Formalism of two-component spinors. Geometrized Heisenberg eqations

We will introduce the two-component spinors o^α and i^α [16], connected with the components of the spinor dyad ξ^α_β as follows:

$$\xi^\alpha_0 = o^\alpha, \quad \xi^\alpha_1 = \iota^\alpha, \quad \overline{\xi}^{\dot{\alpha}}_{\dot{0}} = \overline{o}^{\dot{\alpha}}, \overline{\xi}^{\dot{\alpha}}_{\dot{1}} = \overline{\iota}^{\dot{\alpha}},$$

$$\alpha, \beta\ldots = 0, 1, \quad \dot{\alpha}, \dot{\beta}\ldots = \dot{0}, \dot{1}.$$

From the orthogonality condition for the spinor dyad

$$\begin{aligned}
\xi^0_\alpha\xi^\alpha_1 &= 1, \\
\xi^\alpha_0\xi^0_\alpha = -\xi^0_\alpha\xi^\alpha_0 &= 0, \\
\xi^\alpha_1\xi^1_\alpha &= 0.
\end{aligned} \tag{225}$$

$$\begin{aligned}
\xi^0_\alpha\xi^\beta_0 - \xi^1_\alpha\xi^\beta_0 &= \delta^\beta_\alpha, \\
\xi^0_\alpha\xi^1_\beta - \xi^1_\alpha\xi^0_\beta &= \varepsilon_{\alpha\beta},
\end{aligned} \tag{226}$$

where

$$\varepsilon_{\alpha\beta} = \varepsilon^{\alpha\beta} = \varepsilon_{\dot{\gamma}\dot{\delta}} = \varepsilon^{\dot{\gamma}\dot{\delta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{227}$$

we derive the normalization condition for the two-component spinors

$$\begin{aligned}
o_\alpha\iota^\alpha = -\iota_\alpha o^\alpha &= 1, \\
o^\alpha o_\alpha = -o_\alpha o^\alpha = 0, \quad \iota^\alpha\iota_\alpha &= 0,
\end{aligned} \tag{228}$$

and also the relationships

$$\varepsilon^{\alpha\beta} = o^\alpha\iota^\beta - \iota^\alpha o^\beta, \quad \varepsilon_{\alpha\beta} = o_\alpha\iota_\beta - o_\beta\iota_\alpha, \quad \varepsilon^\beta_\alpha = o_\alpha\iota^\beta - \iota_\alpha o^\beta.$$

Spinors o^α and i^β define the components of the Newman-Penrose symbols (6)

$$\sigma^i_{\ A\dot{B}} = \sigma^i_{\ \alpha\dot{\beta}}\xi^\alpha_A\overline{\xi}^{\dot{\beta}}_{\dot{B}} \tag{229}$$

as follows:

$$\begin{aligned}
\sigma^i_{\ 0\dot{0}} = \sigma^i_{\ \alpha\dot{\beta}}o^\alpha\overline{o}^{\dot{\beta}} = l^i, \quad \sigma^i_{\ 1\dot{1}} = \sigma^i_{\ \alpha\dot{\beta}}\iota^\alpha\overline{\iota}^{\dot{\beta}} = n^i, \\
\sigma^i_{\ 0\dot{1}} = \sigma^i_{\ \alpha\dot{\beta}}o^\alpha\overline{\iota}^{\dot{\beta}} = m^i, \quad \sigma^i_{\ 1\dot{0}} = \sigma^i_{\ \alpha\dot{\beta}}\iota^\alpha\overline{o}^{\dot{\beta}} = \overline{m}^i.
\end{aligned} \tag{230}$$

The vectors l^i, n^i, m^i and $\overline{m}^i$ form an isotropic tetrad. The conventional tetrad ${e^i}_a$ can be made up of the vectors of an isotropic tetrad using the relationships

$$\begin{aligned} e_0^i &= (2)^{-1/2}(l^i + n^i) = (2)^{-1/2}\sigma^i_{\alpha\dot\beta}(o^\alpha\overline{o}^{\dot\beta} + \iota^\alpha\overline{\iota}^{\dot\beta}),\\ e_1^i &= (2)^{-1/2}(m^i + \overline{m}^i) = (2)^{-1/2}\sigma^i_{\alpha\dot\beta}(o^\alpha\overline{\iota}^{\dot\beta} + \iota^\alpha\overline{o}^{\dot\beta}),\\ e_2^i &= (2)^{-1/2}i(m^i - \overline{m}^i) = (2)^{-1/2}\sigma^i_{\alpha\dot\beta}(o^\alpha\overline{\iota}^{\dot\beta} - \iota^\alpha\overline{o}^{\dot\beta}),\\ e_3^i &= (2)^{-1/2}(l^i - n^i) = (2)^{-1/2}\sigma^i_{\alpha\dot\beta}(o^\alpha\overline{o}^{\dot\beta} - \iota^\alpha\overline{\iota}^{\dot\beta}). \end{aligned} \tag{231}$$

Using the relationships

$$T_{ACk} = \frac{1}{2}\varepsilon^{\dot B\dot D}\sigma^i_{C\dot D}\nabla_k\sigma_{A\dot Bi}, \tag{232}$$

$$\nabla_{\alpha\dot\beta} = \sigma^i_{\alpha\dot\beta}\nabla_i \tag{233}$$

we find the following expressions for the components of the Carmeli matrices [17]:

$$\begin{aligned} -\kappa &= o^\alpha\overline{o}^{\dot\beta}o^\gamma\nabla_{\alpha\dot\beta}o_\gamma, & -\lambda &= \iota^\alpha\overline{o}^{\dot\beta}\iota^\gamma\nabla_{\alpha\dot\beta}\iota_\gamma,\\ -\rho &= \iota^\alpha\overline{o}^{\dot\beta}o^\gamma\nabla_{\alpha\dot\beta}o_\gamma, & -\pi &= o^\alpha\overline{o}^{\dot\beta}\iota^\gamma\nabla_{\alpha\dot\beta}\iota_\gamma,\\ -\sigma &= o^\alpha\overline{\iota}^{\dot\beta}o^\gamma\nabla_{\alpha\dot\beta}o_\gamma, & -\varepsilon &= o^\alpha\overline{o}^{\dot\beta}\iota^\gamma\nabla_{\alpha\dot\beta}o_\gamma,\\ -\tau &= \iota^\alpha\overline{\iota}^{\dot\beta}o^\gamma\nabla_{\alpha\dot\beta}o_\gamma, & -\beta &= o^\alpha\overline{\iota}^{\dot\beta}\iota^\gamma\nabla_{\alpha\dot\beta}o_\gamma,\\ -\nu &= \iota^\alpha\overline{\iota}^{\dot\beta}\iota^\gamma\nabla_{\alpha\dot\beta}\iota_\gamma, & -\gamma &= \iota^\alpha\overline{\iota}^{\dot\beta}o^\gamma\nabla_{\alpha\dot\beta}\iota_\gamma,\\ -\mu &= o^\alpha\overline{\iota}^{\dot\beta}\iota^\gamma\nabla_{\alpha\dot\beta}\iota_\gamma, & -\alpha &= \iota^\alpha\overline{o}^{\dot\beta}o^\gamma\nabla_{\alpha\dot\beta}\iota_\gamma, \end{aligned} \tag{234}$$

$$\begin{aligned} \Psi_0 &= \Psi_{\alpha\beta\chi\delta}o^\alpha o^\beta o^\chi o^\delta, & \Psi_1 &= \Psi_{\alpha\beta\chi\delta}o^\alpha o^\beta o^\chi \iota^\delta,\\ \Psi_2 &= \Psi_{\alpha\beta\chi\delta}o^\alpha o^\beta \iota^\chi \iota^\delta, & \Psi_3 &= \Psi_{\alpha\beta\chi\delta}o^\alpha \iota^\beta \iota^\chi \iota^\delta, \end{aligned} \tag{235}$$

$$\Psi_4 = \Psi_{\alpha\beta\chi\delta}\iota^\alpha\iota^\beta\iota^\chi\iota^\delta,$$

$$\begin{aligned} \Phi_{00} &= \overline{\Phi}_{00} = \Phi_{\alpha\beta\dot\chi\dot\delta}o^\alpha o^\beta\overline{o}^{\dot\chi}\overline{o}^{\dot\delta}, & \Phi_{01} &= \overline{\Phi}_{10} = \Phi_{\alpha\beta\dot\chi\dot\delta}o^\alpha o^\beta\overline{o}^{\dot\chi}\overline{\iota}^{\dot\delta},\\ \Phi_{02} &= \overline{\Phi}_{2} = \Phi_{\alpha\beta\dot\chi\dot\delta}o^\alpha o^\beta\overline{\iota}^{\dot\chi}\overline{\iota}^{\dot\delta}, & \Phi_{11} &= \overline{\Phi}_{11} = \Phi_{\alpha\beta\dot\chi\dot\delta}o^\alpha \iota^\beta\overline{o}^{\dot\chi}\overline{\iota}^{\dot\delta},\\ \Phi_{12} &= \overline{\Phi}_{21} = \Phi_{\alpha\beta\dot\chi\dot\delta}o^\alpha \iota^\beta\overline{\iota}^{\dot\chi}\overline{\iota}^{\dot\delta}, & \Phi_{22} &= \overline{\Phi}_{22} = \Phi_{\alpha\beta\dot\chi\dot\delta}\iota^\alpha \iota^\beta\overline{\iota}^{\dot\chi}\overline{\iota}^{\dot\delta}. \end{aligned} \tag{236}$$

It follows from (234) that

$$\begin{aligned} \nabla_{\beta\dot\chi}o_\alpha = \gamma o_\alpha o_\beta\overline{o}_{\dot\chi} - \alpha o_\alpha o_\beta\overline{\iota}_{\dot\chi} - \beta o_\alpha\iota_\beta\overline{o}_{\dot\chi} + \varepsilon o_\alpha\iota_\beta\overline{\iota}_{\dot\chi} -\\ -\tau\iota_\alpha o_\beta\overline{o}_{\dot\chi} + \rho\iota_\alpha o_\beta\overline{\iota}_{\dot\chi} + \sigma\iota_\alpha\iota_\beta\overline{o}_{\dot\chi} - \kappa\iota_\alpha\iota_\beta\overline{\iota}_{\dot\chi}, \end{aligned} \tag{237}$$

$$\begin{aligned} \nabla_{\beta\dot\chi}\iota_\alpha = \nu o_\alpha o_\beta\overline{o}_{\dot\chi} - \lambda o_\alpha o_\beta\overline{\iota}_{\dot\chi} - \mu o_\alpha\iota_\beta\overline{o}_{\dot\chi} + \pi o_\alpha\iota_\beta\overline{\iota}_{\dot\chi} -\\ -\gamma\iota_\alpha o_\beta\overline{o}_{\dot\chi} + \alpha\iota_\alpha o_\beta\overline{\iota}_{\dot\chi} + \beta\iota_\alpha\iota_\beta\overline{o}_{\dot\chi} - \varepsilon\iota_\alpha\iota_\beta\overline{\iota}_{\dot\chi}. \end{aligned} \tag{238}$$

This equations generalizes the nonlinear spinor Heisenberg equations.

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