**A₄ geometry as a group manifold.**

**Killing-Cartan metric**

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The matrix representation of Cartan’s structural equations of the geometry of absolute parallelism indicates that, in fact, this space behaves as a manifold, on which the translations group $T_4$ and the rotations group $O(3.1)$ are specified. We will consider $A_4$ geometry as a group 10-dimensional manifold formed by four translational coordinates $x_i$ $(i = 0, 1, 2, 3)$ and six (by the relationship $e^a_i e^j_a = \delta^j_i$) angular coordinates $e^a_i$ $(a = 0, 1, 2, 3)$. Suppose that on this manifold a group of four-dimensional translations $T_4$ and a rotations group $O(3.1)$ are defined. We then introduce the Hayashi invariant derivative $[1]$

$$\nabla_b = e^k_b \partial_k,$$  \hspace{1cm} (1)

whose components are generators of the translations group $T_4$ that is specified on the manifold of translational coordinates $x_i$. If then we represent as a sum

$$e^k_b = \delta^k_b + a^k_b,$$  \hspace{1cm} (2)

then the field $a^k_b$ can be viewed as the potential of the gauge field of the translations group $T_4$ $[1]$. In the case where $a^k_b = 0$, the generators (1) coincide with the generators of the translations group of the pseudo-Euclidean space $E_4$.

We know already that in the coordinate index $k$ the nonholonomic tetrad $e^k_a$ transforms as the vector

$$e'_a = \frac{\partial x'^k}{\partial x^l} e^k_a,$$  \hspace{1cm} (3)

whence, by (2), we have the law of transformation for the field $a'^k_a$ relative to the translation

$$a'^k_a = \frac{\partial x'^k}{\partial x^l} a'^b_b + \frac{\partial x'^k}{\partial x^l} \delta^a_b - \delta^k_b.$$  \hspace{1cm} (4)

We define the tetrad $e^i_a$ as

$$e^i_a = \nabla_a x^i,$$  \hspace{1cm} (5)

and write the commutational relationships for the generators (1) as

$$\nabla^i_a \nabla_b = -\Omega_{abc} \nabla_c,$$  \hspace{1cm} (6)

where $-\Omega_{abc}$ are the structural functions for the translations group of the space $A_4$. If then we apply the operator (5) to the manifold $x^i$, we will arrive at the structural equations of the group $T_4$ of the space $A_4$ as

$$\nabla^i_a \nabla_b x^i = -\Omega_{abc} \nabla_c x^i.$$  \hspace{1cm} (7)
or

\[ \nabla [a e^i_b] = -\Omega^c_{ab} e^i_c. \]  

(7)

In this relationship the structural functions \(-\Omega^c_{ab}\) are defined as

\[ -\Omega^c_{ab} = e^c_i \nabla [a e^i_b]. \]  

(8)

It is seen from this equality that when the potentials of the gauge field of translations group \(a^k_b\) in the relationship (2) vanish, so do the structural functions (8). Therefore, we will refer to the field \(\Omega^c_{ab}\) as the gauge field of the translations group.

Considering that \(T^c_{[ab]} = -\Omega^c_{ab}\), we will rewrite the structural equations (8) as

\[ \nabla [k e^a_m] - e^b_i [k T^a_{[b|m]}] = 0. \]  

(9)

What is more, they coincide with the structural Cartan equations (A) of the geometry of absolute parallelism.

The structural equations of group \(T_4\), written as (8), can be regarded as a definition for the torsion of space \(A_4\). So the torsion of space \(A_4\) coincides with the structural function of the translations group of this space, such that the structural functions obey the generalized Jacobi identity

\[ \hat{\nabla} \Omega^a_{cd} + 2 \Omega^f_{[bc} \Omega^a_{d]f} = 0, \]  

(10)

where \(\hat{\nabla} \) is the covariant derivative with respect to the connection of absolute parallelism \(\Delta^a_{bc}\). The Jacobi identity (10), which is obeyed by the structural functions of the translations group of geometry \(A_4\), coincides with the first Bianchi identity of the geometry of absolute parallelism.

The vectors

\[ e^i_a = \nabla a x^i, \]  

(11)

that form the vector stratification [1] of the \(A_4\) geometry, point along the tangents to each point of the manifold \(x^i\) of the pseudo-Euclidean plane with the metric tensor

\[ \eta_{ab} = \eta^{ab} = \text{diag}(1, -1, -1, -1). \]  

(12)

Therefore, the ten-dimensional manifold (four translational coordinates \(x^i\) and six "rotational" coordinates \(e^i_a\)) of the geometry of absolute parallelism can be regarded as the stratification with the coordinates of the base \(x^i\) and the (anholonomic) "coordinates" of the fibre \(e^i_a\). If on the base \(x^i\) we have the translations group \(T_4\), then in the fibre \(e^i_a\) we have the rotation group \(O(3,1)\). It follows from (11) that the infinitesimal translations in the base \(x^i\) in the direction \(a\) are given by the vector

\[ ds^a = e^i_a dx^i. \]  

(13)

If from (13) and the covariant vector \(ds_a = e^i_a dx_i\) we form the invariant convolution \(ds^2\), we will obtain the Riemannian metric of \(A_4\) space

\[ ds^2 = g_{ik} dx^i dx^k \]  

(14)

with the metric tensor

\[ g_{ik} = \eta_{ab} e^a_i e^b_k. \]
Therefore, the Riemannian metric (14) can be viewed as the metric defined on the translations group $T_4$.

Since in the fibre we have the "angular coordinates" $e^i_a$ that form a manifold in which group $O(3,1)$ is defined, then it would be natural to define the structural equations for this group, as well as the metric specified on the group $O(3,1)$.

Let us rewrite the Ricci rotational coefficients $T^i_{jk}$ in matrix form

\[ T^a_{bk} = e^a_i T^i_{jk} e^j_b = \nabla_k e^a_j e^j_b, \]  
\[ T^a_{bk} = e^a_i T^i_{jk} e^j_b = -\epsilon^a_i \nabla_k e^j_b. \]  

These relationships enable the dependence between the infinitesimal rotation $d\chi_{ab} = -d\chi_{ba}$ of the vector $e^a_i$ at infinitesimal translations $ds_a$ to be established. In fact, by (15) and (16), we have

\[ d\chi^a_{b} = T^a_{bk} d\chi^b = D e^a_j e^j_b, \]  
\[ d\chi^a_{b} = T^a_{bk} d\chi^b = -\epsilon^a_i \nabla_k e^j_b, \]  

where $D$ is the absolute differential with respect to the Christoffel symbols $\Gamma^i_{jk}$. Using (17), we can form the invariant quadratic form $d\tau^2 = d\chi^a_{(b} d\chi^b_{a)}$ to arrive at the Killing-Cartan metric

\[ d\tau^2 = d\chi^a_{b} d\chi^b_{a} = T^a_{bk} T^b_{an} d\chi^k d\chi^n = -D e^a_i D e^i_a \]  

with the metric tensor

\[ H_{kn} = T^a_{bk} T^b_{an}. \]  

Unlike metric (14), the metric (19) is specified on the rotations group $O(3,1)$ that acts on the manifold of the "rotational coordinates" $e^a_i$.

Let us now introduce the covariant derivative

\[ \nabla_m = \nabla_m + T_m, \]  

where $T_m$ is the matrix $T^a_{bn}$ with discarded matrix indices. We will regard the components of the derivative as generators of the rotations group $O(3,1)$. Applying this operator to the tetrad $e^i$ that forms the manifold of "angular coordinates" of the $A_4$ geometry, we will arrive at

\[ \nabla_m e^i = \nabla_m e^i + T_m e^i = 0, \]  

hence

\[ T_m = -e^i \nabla_m e^i. \]  

It is interesting to note that, just as in (11) we have defined six "angular coordinates" $e^i_a$ through the four translational coordinates $x^i$, so in (5.121) we can define 24 "supercoordinates" $T^a_{bn}$, through the six coordinates $e^i_a$.

It follows from (22) that

\[ \nabla_m e^i = -T_m e^i. \]  

Recall that in the relationships (22)-(24) we have defined through $\nabla_m$ the covariant derivative with respect to $\Gamma^i_{jk}$. We will now take the covariant derivative $\nabla_k$ of the relationships (24)

\[ \nabla_k \nabla_m e^i = -\nabla_k (T_m e^i) = -(\nabla_k T_m e^i + T_m \nabla_k e^i) = 
= -(\nabla_k T_m e^i + T_m e^i \nabla_k e^i). \]
Using (23), we will rewrite this expression as follows

$$\nabla_k \nabla_m e^i = - (\nabla_k T_m - T_m T_k) e^i.$$

Alternating this expression in the indices $k$ and $m$ gives

$$\nabla_{[k} \nabla_{m]} e^i = \frac{1}{2} R_{km} e^i,$$

where

$$R_{km} = 2 \nabla_{[m} T_{k]} + [T_m, T_k].$$

Introducing in equations (26) the matrix indices (the fibre indices), we will obtain the structural equation of the group $O(3.1)$

$$R^a_{bkm} = 2 \nabla_{[m} T^a_{b|k]} + 2 T^a_{c|m} T^c_{b|k]}.$$  \hfill (B)

It is easily seen that the structural equations of the rotations group (B) coincide with the second of Cartan’s structural equations (26) of the geometry $A_4$.

In this case the quantities $T^a_{b|k}$ and $R^a_{bkm}$ vary in the rotations group $O(3.1)$ following the law

$$T^a_{b'k} = \Lambda_a^{a'} T^a_{b'k} \Lambda^b_{b'} + \Lambda_a^{a'} \Lambda^a_{b'k};$$  \hfill (27)

and appear as the potentials of the gauge field $R^a_{bkm}$ of the rotations group $O(3.1)$. In the process, the gauge field of the group $O(3.1)$ obeys the formula

$$R^a_{b'km} = \Lambda_a^{a'} R^a_{bkm} \Lambda^b_{b'}.$$  \hfill (28)

Note that the structural functions of the rotations group of $A_4$ geometry are the components of the curvature tensor $R^a_{bkm}$. It can be shown that the structural functions $R^a_{bkm}$ of the rotations group $O(3.1)$ satisfy the Jacobi identity

$$\nabla_{[n} R^a_{b|km]} + R^c_{b|km} T^a_{c|n]} - T^c_{b|n} R^a_{c|km]} = 0,$$  \hfill (D)

which, at it was shown in the previous section, are at the same time the second Bianchi identities of the $A_4$ space.

Let us introduce the dual Riemann tensor

$$R'_{ijkm} = \frac{1}{2} \varepsilon^{sp}_{km} R_{ijsp},$$  \hfill (29)

where $\varepsilon^{sp}_{km}$ is the completely skew-symmetric Levi-Civita tensor. Then the equations (D) can be written as

$$\nabla_n R^a_{bkn} + R^c_{bkn} T^a_{cn} - T^c_{bn} R^a_{ckn} = 0$$  \hfill (30)

or, if we drop the matrix indices, as

$$\nabla_n \mathring{R}_{kn} + \mathring{R}_{kn} T_n - T_n \mathring{R}_{kn} = 0.$$  \hfill (31)

\textbf{References}