A_4 geometry as a group manifold. Killing-Cartan metric

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The matrix representation of Cartan's structural equations of the geometry of absolute parallelism indicates that, in fact, this space behaves as a manifold, on which the translations group T_4 and the rotations group $O(3.1)$ are specified. We will consider A_4 geometry as a group 10-dimensional manifold formed by four translational coordinates x_i $(i = 0, 1, 2, 3)$ and six (by the relationship $e^a{}_i e^j{}_a = \delta_i{}^j$) angular coordinates $e^a{}_i$ $(a = 0, 1, 2, 3)$. Suppose that on this manifold a group of four-dimensional translations T_4 and a rotations group $O(3.1)$ are defined. We then introduce the Hayashi invariant derivative [1]

$$\nabla_b = e^k{}_b \partial_k, \tag{1}$$

whose components are generators of the translations group T_4 that is specified on the manifold of translational coordinates x_i. If then we represent as a sum

$$e^k{}_b = \delta^k{}_b + a^k{}_b, \tag{2}$$

$$i, j, k \ldots = 0, 1, 2, 3, \quad a, b, c, \ldots = 0, 1, 2, 3,$$

then the field $a^k{}_b$ can be viewed as the potential of the gauge field of the translations group T_4 [1]. In the case where $a^k{}_b = 0$, the generators (1) coincide with the generators of the translations group of the pseudo-Euclidean space E_4 .

We know already that in the coordinate index k the nonholonomic tetrad $e^k{}_a$ transforms as the vector

$$e^{k'}{}_a = \frac{\partial x^{k'}}{\partial x^k} e^k{}_a,$$

whence, by (2), we have the law of transformation for the field $a^k{}_a$ relative to the translationss

$$a^{k'}{}_b = \frac{\partial x^{k'}}{\partial x^n} a^n{}_b + \frac{\partial x^{k'}}{\partial x^n} \delta^n{}_b - \delta^{k'}{}_b. \tag{3}$$

We define the tetrad $e^i{}_a$ as

$$e^i{}_a = \nabla_a x^i \tag{4}$$

and write the commutational relationships for the generators (1) as

$$\nabla_{[a} \nabla_{b]} = -\Omega_{ab}^{\cdot\cdot c} \nabla_c, \tag{5}$$

where $-\Omega_{ab}^{\cdot\cdot c}$ are the structural functions for the translations group of the spaceA_4. If then we apply the operator (5) to the manifold x^i, we will arrive at the structural equations of the group T_4 of the space A_4 as

$$\nabla_{[a} \nabla_{b]} x^i = -\Omega_{ab}^{\cdot\cdot c} \nabla_c x^i \tag{6}$$

or

$$\nabla_{[a}e^{i}{}_{b]} = -\Omega^{..c}_{ab}e^{i}{}_{c}. \tag{7}$$

In this relationship the structural functions $-\Omega^{..c}_{ab}$ are defined as

$$-\Omega^{..c}_{ab} = e^{c}{}_{i}\nabla_{[a}e^{i}{}_{b]}. \tag{8}$$

It is seen from this equality that when the potentials of the gauge field of translations group a^{k}_{b} in the relationship (2) vanish, so do the structural functions (8). Therefore, we will refer to the field $\Omega^{..c}_{ab}$ as the gauge field of the translations group.

Considering that $T^{c}{}_{[ab]} = -\Omega^{..c}_{ab}$, we will rewrite the structural equations (8) as

$$\nabla_{[k}e^{a}{}_{m]} - e^{b}{}_{[k}T^{a}{}_{|b|m]} = 0. \tag{9}$$

What is more, they coincide with the structural Cartan equations (A) of the geometry of absolute parallelism.

The structural equations of group T_4, written as (8), can be regarded as a definition for the torsion of space A_4. So the torsion of space A_4 coincides with the structural function of the translations group of this space, such that the structural functions obey the generalized Jacobi identity

$$\overset{*}{\nabla}_{[b}\,\Omega^{..a}_{cd]} + 2\Omega^{..f}_{[bc}\Omega^{..a}_{d]f} = 0, \tag{10}$$

where $\overset{*}{\nabla}_{b}$ is the covariant derivative with respect to the connection of absolute parallelism Δ^{a}_{bc}. The Jacobi identity (10), which is obeyed by the structural functions of the translations group of geometry A_4, coincides with the first Bianchi identity of the geometry of absolute parallelism .

The vectors

$$e^{i}{}_{a} = \nabla_{a}x^{i}, \tag{11}$$

that form the vector stratification [1] of the A_4 geometry, point along the tangents to each point of the manifold x^i of the pseudo-Euclidean plane with the metric tensor

$$\eta_{ab} = \eta^{ab} = \mathrm{diag}(1,\,-1,\,-1,\,-1). \tag{12}$$

Therefore, the ten-dimensional manifold (four translational coordinates x^i and six "rotational" coordinates $e^{i}{}_{a}$) of the geometry of absolute parallelism can be regarded as the stratification with the coordinates of the base x^i and the (anholonomic) "coordinates" of the fibre $e^{i}{}_{c}$. If on the base x^i we have the translations group T_4, then in the fibre $e^{i}{}_{c}$ we have the rotation group $O(3.1)$. It follows from (11) that the infinitesimal translations in the base x^i in the direction a are given by the vector

$$ds^{a} = e^{a}{}_{i}dx^{i}. \tag{13}$$

If from (13) and the covariant vector $ds_a = e^{i}{}_{a}dx_i$ we form the invariant convolution ds^2, we will obtain the Riemannian metric of A_4 space

$$ds^{2} = g_{ik}dx^{i}dx^{k} \tag{14}$$

with the metric tensor

$$g_{ik} = \eta_{ab}e^{a}{}_{i}e^{b}{}_{k}.$$

Therefore, the Riemannian metric (14) can be viewed as the metric defined on the translations group T_4.

Since in the fibre we have the "angular coordinates" $e^i{}_a$ that form a manifold in which group $O(3.1)$ is defined, then it would be natural to define the structural equations for this group, as well as the metric specified on the group $O(3.1)$.

Let us rewrite the Ricci rotational coefficients $T^i{}_{jk}$ in matrix form

$$T^a_{\ bk} = e^a{}_i T^i{}_{jk} e^j{}_b = \nabla_k e^a{}_j e^j{}_b, \tag{15}$$

$$T^a_{\ bk} = e^a{}_i T^i{}_{jk} e^j{}_b = -e^a{}_i \nabla_k e^i{}_b. \tag{16}$$

These relationships enable the dependence between the infinitesimal rotation $d\chi_{ab} = -d\chi_{ba}$ of the vector $e^a{}_i$ at infinitesimal translations ds_a to be established. In fact, by (15) and (16), we have

$$d\chi^a{}_b = T^a_{\ bk} dx^k = De^a{}_j e^j{}_b, \tag{17}$$

$$d\chi^a{}_b = T^a_{\ bk} dx^k = -e^a{}_i De^i{}_b. \tag{18}$$

where D is the absolute differential with respect to the Christoffel symbols Γ^i_{jk}. Using (17), we can form the invariant quadratic form $d\tau^2 = d\chi^a{}_b d\chi^b{}_a$ to arrive at the Killing-Cartan metric

$$d\tau^2 = d\chi^a{}_b d\chi^b{}_a = T^a_{\ bk} T^b{}_{an} dx^k dx^n = -De^a{}_i De^i{}_a \tag{19}$$

with the metric tensor

$$H_{kn} = T^a_{\ bk} T^b{}_{an}. \tag{20}$$

Unlike metric (14), the metric (19) is specified on the rotations group $O(3.1)$ that acts on the manifold of the "rotational coordinates" $e^a{}_i$.

Let us now introduce the covariant derivative

$$\overset{*}{\nabla}_m = \nabla_m + T_m, \tag{21}$$

where T_m is the matrix $T^a_{\ bm}$ with discarded matrix indices. We will regard the components of the derivative as generators of the rotations group $O(3.1)$. Applying this operator to the tetrad e^i that forms the manifold of "angular coordinates" of the A_4 geometry, we will arrive at

$$\overset{*}{\nabla}_m e^i = \nabla_m e^i + T_m e^i = 0, \tag{22}$$

hence

$$T_m = -e_i \nabla_m e^i. \tag{23}$$

It is interesting to note that, just as in (11) we have defined six "angular coordinates" $e^i{}_a$ through the four translational coordinates x^i, so in (5.121) we can define 24 "supercoordinates" $T^a_{\ bm}$ through the six coordinates $e^i{}_a$.

It follows from (22) that

$$\nabla_m e^i = -T_m e^i. \tag{24}$$

Recall that in the relationships (22)-(24) we have defined through ∇_m the covariant derivative with respect to Γ^i_{jk}. We will now take the covariant derivative ∇_k of the relationships (24)

$$\nabla_k \nabla_m e^i = -\nabla_k (T_m e^i) = -(\nabla_k T_m e^i + T_m \nabla_k e^i) =$$
$$= -(\nabla_k T_m e^i + T_m e^i e_i \nabla_k e^i).$$

Using (23), we will rewrite this expression as follows

$$\nabla_k \nabla_m e^i = -(\nabla_k T_m - T_m T_k) e^i.$$

Alternating this expression in the indices k and m gives

$$\nabla_{[k} \nabla_{m]} e^i = \frac{1}{2} R_{km} e^i, \tag{25}$$

where

$$R_{km} = 2\nabla_{[m} T_{k]} + [T_m,\, T_k]. \tag{26}$$

Introducing in equations (26) the matrix indices (the fibre indices), we will obtain the structural equation of the group $O(3.1)$

$$R^a{}_{bkm} = 2\nabla_{[m} T^a{}_{|b|k]} + 2T^a{}_{c[m} T^c{}_{|b|k]}. \tag{B}$$

It is easily seen that the structural equations of the rotations group (B) coincide with the second of Cartan's structural equations (26) of the geometry A_4.

In this case the quantities T^a_{bk} and $R^a{}_{bkm}$ vary in the rotations group $O(3.1)$ following the law

$$T^{a'}_{b'k} = \Lambda_a{}^{a'} T^a_{bk} \Lambda^b{}_{b'} + \Lambda_a{}^{a'} \Lambda^a{}_{b',k}, \tag{27}$$

and appear as the potentials of the gauge field $R^a{}_{bkm}$ of the rotations group $O(3.1)$. In the process, the gauge field of the group $O(3.1)$ obeys the formula

$$R^{a'}_{b'km} = \Lambda_a{}^{a'} R^a{}_{bkm} \Lambda^b{}_{b'}. \tag{28}$$

Note that the structural functions of the rotations group of A_4 geometry are the components of the curvature tensor $R^a{}_{bkm}$. It can be shown that the structural functions $R^a{}_{bkm}$ of the rotations group $O(3.1)$ satisfy the Jacobi identity

$$\nabla_{[n} R^a{}_{|b|km]} + R^c{}_{b[km} T^a{}_{|c|n]} - T^c{}_{b[n} R^a{}_{|c|km]} = 0, \tag{D}$$

which, at it was shown in the previous section, are at the same time the second Bianchi identities of the A_4 space.

Let us introduce the dual Riemann tensor

$$\overset{*}{R}_{ijkm} = \frac{1}{2} \varepsilon^{sp}{}_{km} R_{ijsp}, \tag{29}$$

where $\varepsilon^{sp}{}_{km}$ is the completely skew-symmetrical Levi-Chivita tensor. Then the equations (D) can be written as

$$\nabla_n \overset{*}{R}{}^a{}_b{}^{kn} + \overset{*}{R}{}^c{}_b{}^{kn} T^a{}_{cn} - T^c{}_{bn} \overset{*}{R}{}^a{}_c{}^{kn} = 0 \tag{30}$$

or, if we drop the matrix indices, as

$$\nabla_n \overset{*}{R}{}^{kn} + \overset{*}{R}{}^{kn} T_n - T_n \overset{*}{R}{}^{kn} = 0. \tag{31}$$

References

[1] *Hayashi K.* // Phys. Lett. B. 1977. Vol. 69, N 4, pp. 441-443.