# A history of appearance of Cartan torsion in the differential geometry


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Geometry with absolute parallelism was first considered in 1923-24 in the works of Weitzenbock [1, 2] and Vitali [3, 4]. Weitzenbock suggested that there exist in the $n$-dimensional manifold $M$ with coordinates $x^1, \ldots, x^n$ of Riemannian spaces with a zero Riemann-Christoffel tensor

$$S^i{}_{jkm} = 2\Delta^i_{j[m,k]} + 2\Delta^i_{s[k}\Delta^s_{|j|m]} = 0. \tag{1}$$

Relationship (1) was regarded as the condition of parallel displacement of an arbitrary vector in a given space in the absolute (independent of path) sense. In 1924 Vitali introduced the concepts of the connection of absolute parallelism [3]

$$\Delta^k_{ij} = e^k{}_a e^a{}_{i,j}, \tag{2}$$

$$, j = \frac{\partial}{\partial x^j}, \quad \begin{array}{l} i, j, k \ldots = 0, 1, 2, 3, \\ a, b, c \ldots = 0, 1, 2, 3, \end{array}$$

where $e^k{}_a$ and $e^a{}_i$ are basic vectors defined at each point of space and translatable in the absolute sense to any point of the space in any direction. Weitzenbock [5] showed that the connection (2) can be represented as the sum

$$\Delta^i_{jk} = \Gamma^i_{jk} + T^i_{jk}, \tag{3}$$

where

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m}), \tag{4}$$

are the Christoffel symbols and

$$T^i_{jk} = -\Omega^{..i}_{jk} + g^{im}(g_{js}\Omega^{s}_{mk} + g_{ks}\Omega^{..s}_{mj}) \tag{5}$$

are the Ricci rotation coefficients [6] for the basis $e^a{}_i$.

The tensor $\Omega^{..i}_{jk}$, defined as

$$\Omega^{..i}_{jk} = e^i{}_a e^a_{[k,j]} = \frac{1}{2} e^i{}_a (e^a{}_{k,j} - e^a{}_{j,k}), \tag{6}$$

came to be known as the anholonomity object [7], therefore the emergence of the geometry of absolute parallelism continued the development of anholonomic differential geometry [8].

Cartan and Schouten [9, 10], proceeding from the group properties of the space of constant curvature, introduced the connection (3), in which the components of the Ricci rotation coefficients (5) are constants.

Cartan and Schouten reasoned as follows. Suppose that in a $n$ - dimensional differentiable manifold $M$ with the coordinates $x^1, \ldots, x^n$ we have $n$ contravariant vector fields

$$\xi^j_a = \xi^j_a(x^k), \tag{7}$$

where

$$a, b, c \ldots = 1 \ldots n$$

are vector indices, and

$$i, j, k \ldots = 1 \ldots n$$

are coordinate indices.

Suppose that

$$\det(\xi^j_a) \neq 0$$

and that the functions $\xi^j_a$ satisfy the equations

$$\xi^j_a \xi^k{}_{b,j} - \xi^i_b \xi^k{}_{a,j} = -C^{..f}_{ab} \xi^k_f,$$

where the constants $C^{..f}_{ab}$ have the following properties:

$$C^{..f}_{ab} = -C^{..f}_{ba}, \tag{8}$$

$$C^{..a}_{fb} C^{..f}_{cd} + C^{..a}_{fc} C^{..f}_{db} + C^{..a}_{fd} C^{..f}_{bc} = 0. \tag{9}$$

We can then say that we have an $n$-parametric simple transitive group (group $T_n$) operating in the manifold such that $C^{..f}_{ab}$ are structural constants of the group that obey the Jacobi identity (9). The vector field $\xi^j_b$ is said to be infinitesimal generators of the group.

Let now the basis $e^k_b$, defined at each point of the manifold $M$, meet the condition

$$\det(e^j{}_a) \neq 0.$$

If we suppose that

$$e^j{}_a(x^k_0) = \xi^j_a(x^k_0),$$

where $x^k_0$ are the coordinates of some arbitrary point $P$, then we have for the function $e^j{}_a(x^k_0)$ the equations

$$e^j{}_a e^k{}_{b,j} - e^j{}_b e^k{}_{a,j} = -C^{..f}_{ab} e^k{}_f. \tag{10}$$

It follows from the normalization condition for the basis

$$e^a{}_i e^j{}_a = \delta^j_i, \quad e^a{}_i e^i{}_b = \delta^a_b, \tag{11}$$

and from (10), that

$$C^{..i}_{jk} = 2e^i{}_a e^a{}_{[k,j]} = e^i{}_a C^{..a}_{bc} e^b_j e^c_k. \tag{12}$$

Comparing (8) and (6), we see that

$$\Omega^{..i}_{jk} = \frac{1}{2} C^{..i}_{jk},$$

i.e., the components of the anholonomity object of a homogeneous space of absolute parallelism are constant.

It is easily seen that the connection (2) possesses a torsion. In our specific case

$$\Delta^{k}_{[ij]} = -\Omega^{..k}_{ij} = T^{k}_{[ij]} = -\frac{1}{2}C^{..i}_{jk}.$$

It was exactly in this manner that Cartan and Schouten introduced connection with torsion [9, 10]. Therefore, the development of the geometry of absolute parallelism brought about the emergence of the Riemann-Cartan geometry with the connection

$$\widetilde{\Gamma}_{ijk} = \Gamma_{ijk} + \frac{1}{2}(C_{ijk} - C_{jki} - C_{kij}), \tag{13}$$

where $S_{ijk} = -\frac{1}{2}C_{ijk}$ is the torsion of space.

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