FORMALISM OF EXTERNAL FORMS, EINSTEIN-YANG-MILLS SET OF EQUATIONS AND GAUGE GROUP APPROACH IN A_4 GEOMETRY

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1 Formalism of external forms and the matrix treatment of Cartan's structural equations of the absolute parallelism geometry

Consider the differentials

$$dx^i = e^a e^i{}_a, \tag{1}$$

$$de^i{}_b = \Delta^a{}_b e^i{}_a, \tag{2}$$

where

$$e^a = e^a_i dx^i, \tag{3}$$

$$\Delta^a{}_b = e^a{}_i de^i{}_b = \Delta^a{}_{bk} dx^k \tag{4}$$

are differential 1-forms of tetrad $e^a{}_i$ and connection of absolute parallelism $\Delta^a{}_{bk}$. Differentiating the relationships (1), (2) externally [31], we have, respectively,

$$d(dx^i) = (de^a - e^c \wedge \Delta^a{}_c)e^i{}_a = -S^a e^i{}_a, \tag{5}$$

$$d(de^i{}_a) = (d\Delta^b{}_a - \Delta^c{}_a \wedge \Delta^b{}_c)e^i{}_b = -S^b{}_a e^i{}_b. \tag{6}$$

Here S^a denotes the 2-form of Cartanian torsion [31], and $S^b{}_a$ – the 2-form of the curvature tensor. The sign $\wedge$ signifies external product, e.g,

$$e^a \wedge e^b = e^a e^b - e^b e^a. \tag{7}$$

By definition, a space has a geometry of absolute parallelism, if the 2-form of Cartanian torsion S^a and the 2-form of the Riemann-Christoffel curvature $S^b{}_a$ of this space vanishe

$$S^a = 0, \tag{8}$$

$$S^b{}_a = 0. \tag{9}$$

At the same time, these equalities are the integration conditions for the differentials (1) and (2).

Equations

$$de^a - e^c \wedge \Delta^a{}_c = -S^a, \tag{10}$$

$$d\Delta^b{}_a - \Delta^c{}_a \wedge \Delta^b{}_c = -S^b{}_a, \tag{11}$$

which follow from (5) and (6), are Cartan's structural equations for an appropriate geometry. For the geometry of absolute parallelism hold the conditions (8) and (9), therefore Cartan's structural equations for A_4 geometry have the form

$$de^a - e^c \wedge \Delta^a{}_c = 0, \tag{12}$$

$$d\Delta^b{}_a - \Delta^c{}_a \wedge \Delta^b{}_c = 0. \tag{13}$$

Considering (??), we will represent 1-form $\Delta^a{}_b$ as the sum

$$\Delta^a{}_b = \Gamma^a{}_b + T^a{}_b. \tag{14}$$

Substituting this relationship into (12) and noting that

$$e^c \wedge \Delta^a{}_c = e^c \wedge T^a{}_c,$$

we get the first of Cartan's structural equations for A_4 space.

$$de^a - e^c \wedge T^a{}_c = 0. \tag{A}$$

Substituting (14) into (13) gives the second of Cartan's equations for A_4 space.

$$R^a_b + dT^a{}_b - T^c{}_b \wedge T^a{}_c = 0, \tag{B}$$

where $R^a{}_b$ is the 2-form of the Riemann tensor

$$R^a{}_b = d\Gamma^a{}_b - \Gamma^c{}_b \wedge \Gamma^a{}_c. \tag{15}$$

By definition [31], we always have the relationships

$$dd(dx^i) = 0, \tag{16}$$

$$dd(de^i{}_a) = 0. \tag{17}$$

In the geometry of absolute parallelism these equalities become

$$d(de^a - e^c \wedge T^a{}_c) = R^a{}_{cfd} e^c \wedge e^f \wedge e^d = 0, \tag{18}$$

$$d(R^a{}_b + dT^a{}_b - T^c{}_b \wedge T^a{}_c) = dR^a{}_b + R^f{}_b \wedge T^a{}_f - T^f{}_b \wedge R^a{}_f = 0. \tag{19}$$

Here

$$R^a{}_{cfd} = -2T^a{}_{c[d,f]} - 2T^a{}_{b[f} T^b{}_{|c|d]}.$$

Equalities (18) and (19) represent the first and second of Bianchi's identities, respectively, for A_4 space. Dropping the indices, we can write Cartan's structural equations and Bianchi's identities for the A_4 geometry as

$$\boxed{\begin{array}{lr} de - e \wedge T = 0, & (A) \\ R + dT - T \wedge T = 0, & (B) \\ R \wedge e \wedge e \wedge e = 0, & (C) \\ dR + R \wedge T - T \wedge R = 0. & (D) \end{array}}$$

Proposition 5.8. The matrix treatment of the first of Cartan's structural equations (A) of the A_4 geometry has the form

$$\nabla_{[k} e^a{}_{m]} - e^b{}_{[k} T^a{}_{|b|m]} = 0. \tag{20}$$

Proof. Let us write equations (A) as

$$de^a - e^c \wedge T^a_{\ c} = 0. \tag{21}$$

Further, by (3), we have

$$de^a = d(e^a_{\ m} dx^m) = \nabla_k e^a_{\ m} dx^k \wedge dx^m = \frac{1}{2}(\nabla_k e^a_{\ m} - \nabla_m e^a_{\ k}) dx^k \wedge dx^m$$

and, also,

$$e^b \wedge T^a_{\ b} = e^b_{\ k} T^a_{\ bm} dx^k \wedge dx^m = \frac{1}{2}(e^b_{\ k} T^a_{\ bm} - e^b_{\ m} T^a_{\ bk}) dx^k \wedge dx^m.$$

Substituting these relationships into equations (21) we will derive the matrix equations in the form

$$\nabla_{[k} e^a_{\ m]} - e^b_{\ [k} T^a_{\ |b|m]} = 0, \tag{A}$$

where the matrixes $e^a_{\ m}$ and $T^a_{\ bm}$ in world indices $i, j, m, \dots$ are transformed as vectors

$$e^a_{\ m'} = \frac{\partial x^m}{\partial x^{m'}} e^a_{\ m}, \tag{22}$$

$$T^a_{\ bm'} = \frac{\partial x^m}{\partial x^{m'}} T^a_{\ bm}, \tag{23}$$

and in the matrix indices $a, b, c, \dots$ they are transformed as follows:

$$e^{a'}_{\ m} = \Lambda_a^{\ a'} e^a_{\ m}, \tag{24}$$

$$T^{a'}_{\ b'k} = \Lambda_a^{\ a'} T^a_{\ bk} \Lambda^b_{\ b'} + \Lambda^{a'}_{\ a} \Lambda^a_{\ b',k}. \tag{25}$$

In relationships (24) and (25) the matrices $\partial x^{m'} / \partial x^m$ form a translation group T_4 that is defined on a manifold of world coordinates x^i. On the other hand, the matrices $\Lambda^{a'}_{\ a}$ form a group of four-dimensional rotations $O(3.1)$

$$\Lambda^{a'}_a \in O(3.1),$$

defined on the manifold of "angular coordinates"$e^a_{\ i}$. Actually, the tetrad $e^a_{\ i}$ is a mathematical image of an arbitrarily accelerated four-dimensional reference frame. Such a frame has ten degrees of freedom: four translational ones connected with the motion of its origin, and six angular ones describing variations of its orientation. The six independent components of the tetrad $e^a_{\ i}$ represent six direction cosines of six independent angles defining the orientation of the tetrad in space.

Proposition 5.9. The matrix rendering of the second of Cartan's structuring equations (B) of the A_4 geometry has the form

$$R^a_{\ bkm} + 2\nabla_{[k} T^a_{\ |b|m]} + 2T^a_{\ c[k} T^c_{\ |b|m]} = 0. \tag{26}$$

Proof. We will expand the 2-form $R^a_{\ d}$ as

$$R^a_{\ b} = \frac{1}{2} R^a_{\ bcd} e^c \wedge e^d = \frac{1}{2} R^a_{\ bkm} dx^k \wedge dx^m. \tag{27}$$

Further, we have

$$dT^a_{\ b} = d(T^a_{\ bm} dx^m) = \nabla_k T^a_{\ bm} dx^k \wedge dx^m = \frac{1}{2}(\nabla_k T^a_{\ bm} - \nabla_m T^a_{\ bk}) dx^k \wedge dx^m, \tag{28}$$

and also

$$T^a_{\ c} \wedge T^c_{\ b} = T^a_{\ ck} T^c_{\ bm} dx^k \wedge dx^m = = \frac{1}{2}(T^a_{\ ck} T^c_{\ bm} - T^c_{\ bm} T^a_{\ ck}) dx^k \wedge dx^m. \tag{29}$$

Let us substitute the relationships (29)–(31) into

$$R^a_{\ b} + dT^a_{\ b} - T^c_{\ b} \wedge T^a_{\ c} = 0.$$

Simple transformations yield

$$\frac{1}{2}(R^a_{\ bkm} + \nabla_k T^a_{\ bm} - \nabla_m T^a_{\ bk} + T^a_{\ ck} T^c_{\ bm} - T^c_{\ bm} T^a_{\ ck}) dx^k \wedge dx^m = 0.$$

Since here the factor $dx^k \wedge dx^m$ is arbitrary, we have

$$R^a_{\ bkm} + \nabla_k T^a_{\ bm} - \nabla_m T^a_{\ bk} + T^a_{\ ck} T^c_{\ bm} - T^c_{\ bm} T^a_{\ ck} = 0,$$

which is equivalent to the equations (26).

Proposition 5.10. The matrix form of the Bianchi identity (D) of A_4 geometry is

$$\nabla_{[n} R^a_{\ |b|km]} + R^c_{\ b[km} T^a_{\ |c|n]} - T^c_{\ b[n} R^a_{\ |c|km]} = 0. \tag{30}$$

Proof. The external differential $dR^a{}_b$ in the identities (D) has the 2-form

$$dR^a_{\ b} = \frac{1}{2}\nabla_n R^a_{\ bkm} dx^n \wedge dx^k \wedge dx^m = = \frac{1}{6}(\nabla_n R^a_{\ bkm} + \nabla_m R^a_{\ bkn} + \nabla_k R^a_{\ bmn}) dx^n \wedge dx^k \wedge dx^m. \tag{31}$$

In addition, we have

$$R^f_{\ b} \wedge T^a_{\ f} = \frac{1}{2} R^f_{\ bkm} T^a_{\ fn} dx^k \wedge dx^m \wedge dx^n = = \frac{1}{6}(R^f_{\ bkm} T^a_{\ fn} + R^f_{\ bnk} T^a_{\ fm} + R^f_{\ bmn} T^a_{\ fk}) dx^k \wedge dx^m \wedge dx^n, \tag{32}$$

$$T^f_{\ b} \wedge R^a_{\ f} = \frac{1}{2} T^f_{\ bn} R^a_{\ fkm} dx^n \wedge dx^k \wedge dx^m = = \frac{1}{6}(T^f_{\ bn} R^a_{\ fkm} + T^f_{\ bm} R^a_{\ fnk} + T^f_{\ bk} R^a_{\ fmn}) dx^n \wedge dx^k \wedge dx^m. \tag{33}$$

Substituting relationships (31)–(33) into the identity

$$dR^a_{\ b} + R^f_{\ b} \wedge T^a_{\ f} - T^f_{\ b} \wedge R^a_{\ f} = 0$$

and considering that $dx^n \wedge d^k \wedge dx^m$ is arbitrary, we get

$$\nabla_n R^a{}_{bkm} + \nabla_m R^a{}_{bkn} + \nabla_k R^a{}_{bmn} + R^f{}_{bkm} T^a{}_{fn} + R^f{}_{bnk} T^a{}_{fm} + \\ + R^f{}_{bmn} T^a{}_{fk} - T^f{}_{bn} R^a{}_{fkm} - T^f{}_{bm} R^a{}_{fnk} - T^f{}_{bk} R^a{}_{fmn} = 0,$$

which is equivalent to the identity (30).

The first of Bianchi's identities (C) of A_4 geometry in indices of the group $O(3.1)$ is written as

$$R^a{}_{[bcd]} = 0, \tag{34}$$

or, which is the same, as

$$\overset{*}{\nabla}_{[b}\, \Omega^{..a}_{cd]} + 2\Omega^{..f}_{[bc}\Omega^{..a}_{d]f} = 0. \tag{35}$$

2 A$_4$ geometry as a group manifold. Killing-Cartan metric

The matrix representation of Cartan's structural equations of the geometry of absolute parallelism indicates that, in fact, this space behaves as a manifold, on which the translations group T_4 and the rotations group $O(3.1)$ are specified. We will consider A_4 geometry as a group 10-dimensional manifold formed by four translational coordinates x_i $(i = 0,1,2,3)$ and six (by the relationship $e^a{}_i e^j{}_a = \delta_i{}^j$) angular coordinates $e^a{}_i$ $(a = 0,1,2,3)$. Suppose that on this manifold a group of four-dimensional translations T_4 and a rotations group $O(3.1)$ are defined. We then introduce the Hayashi invariant derivative [32]

$$\nabla_b = e^k{}_b \partial_k, \tag{36}$$

whose components are generators of the translations group T_4 that is specified on the manifold of translational coordinates x_i. If then we represent as a sum

$$e^k{}_b = \delta^k{}_b + a^k{}_b, \tag{37}$$

$$i, j, k \ldots = 0,1,2,3, \quad a, b, c, \ldots = 0,1,2,3,$$

then the field $a^k{}_b$ can be viewed as the potential of the gauge field of the translations group T_4 [32]. In the case where $a^k{}_b = 0$, the generators (36) coincide with the generators of the translations group of the pseudo-Euclidean space E_4 .

We know already that in the coordinate index k the nonholonomic tetrad $e^k{}_a$ transforms as the vector

$$e^{k'}{}_a = \frac{\partial x^{k'}}{\partial x^k} e^k{}_a,$$

whence, by (37), we have the law of transformation for the field $a^k{}_a$ relative to the translationss

$$a^{k'}{}_b = \frac{\partial x^{k'}}{\partial x^n} a^n{}_b + \frac{\partial x^{k'}}{\partial x^n} \delta^n{}_b - \delta^{k'}_b. \tag{38}$$

We define the tetrad $e^i{}_a$ as

$$e^i{}_a = \nabla_a x^i \tag{39}$$

and write the commutational relationships for the generators (36) as

$$\nabla_{[a}\nabla_{b]} = -\Omega_{ab}^{\cdot\cdot c}\nabla_c, \tag{40}$$

where $-\Omega_{ab}^{\cdot\cdot c}$ are the structural functions for the translations group of the spaceA_4. If then we apply the operator (40) to the manifold x^i, we will arrive at the structural equations of the group T_4 of the space A_4 as

$$\nabla_{[a}\nabla_{b]}x^i = -\Omega_{ab}^{\cdot\cdot c}\nabla_c x^i \tag{41}$$

or

$$\nabla_{[a}e^i{}_{b]} = -\Omega_{ab}^{\cdot\cdot c}e^i{}_c. \tag{42}$$

In this relationship the structural functions $-\Omega_{ab}^{\cdot\cdot c}$ are defined as

$$-\Omega_{ab}^{\cdot\cdot c} = e^c{}_i\nabla_{[a}e^i{}_{b]}. \tag{43}$$

It is seen from this equality that when the potentials of the gauge field of translations group a_b^k in the relationship (37) vanish, so do the structural functions (43). Therefore, we will refer to the field $\Omega_{ab}^{\cdot\cdot c}$ as the gauge field of the translations group.

Considering that $T^c{}_{[ab]} = -\Omega_{ab}^{\cdot\cdot c}$, we will rewrite the structural equations (43) as

$$\nabla_{[k}e^a{}_{m]} - e^b{}_{[k}T^a{}_{|b|m]} = 0. \tag{44}$$

It is easily seen that the equations (44) can be derived by alternating the equations (**??**). What is more, they coincide with the structural Cartan equations (A) of the geometry of absolute parallelism.

The structural equations of group T_4, written as (43), can be regarded as a definition for the torsion of space A_4. So the torsion of space A_4 coincides with the structural function of the translations group of this space, such that the structural functions obey the generalized Jacobi identity

$$\overset{*}{\nabla}_{[b}\,\Omega_{cd]}^{\cdot\cdot a} + 2\Omega_{[bc}^{\cdot\cdot f}\Omega_{d]f}^{\cdot\cdot a} = 0, \tag{45}$$

where $\overset{*}{\nabla}_b$ is the covariant derivative with respect to the connection of absolute parallelism Δ^a_{bc}. Comparing the identity (45) with the Bianchi identity (35) of the geometry A_4, we see that we deal with the same identity. The Jacobi identity (5.108), which is obeyed by the structural functions of the translations group of geometry A_4, coincides with the first Bianchi identity of the geometry of absolute parallelism .

The vectors

$$e^i{}_a = \nabla_a x^i, \tag{46}$$

that form the vector stratification [31] of the A_4 geometry, point along the tangents to each point of the manifold x^i of the pseudo-Euclidean plane with the metric tensor

$$\eta_{ab} = \eta^{ab} = \mathrm{diag}(1,\, -1,\, -1,\, -1). \tag{47}$$

Therefore, the ten-dimensional manifold (four translational coordinates x^i and six "rotational"coordinates $e^i{}_a$) of the geometry of absolute parallelism can be regarded as the stratification with the coordinates of the base x^i and the (anholonomic) "coordinates"of the fibre ${e^i}_c$. If on the base x^i we have the translations group T_4, then in the fibre ${e^i}_c$ we have the rotation group $O(3.1)$. It follows from (46) that the infinitesimal translations in the base x^i in the direction a are given by the vector

$$ds^a = {e^a}_i dx^i. \tag{48}$$

If from (48) and the covariant vector $ds_a = {e^i}_a dx_i$ we form the invariant convolution ds^2, we will obtain the Riemannian metric of A_4 space

$$ds^2 = g_{ik} dx^i dx^k \tag{49}$$

with the metric tensor

$$g_{ik} = \eta_{ab} {e^a}_i {e^b}_k.$$

Therefore, the Riemannian metric (49) can be viewed as the metric defined on the translations group T_4.

Since in the fibre we have the "angular coordinates"${e^i}_a$ that form a manifold in which group $O(3.1)$ is defined, then it would be natural to define the structural equations for this group, as well as the metric specified on the group $O(3.1)$.

Let us rewrite the relationships (**??**) and (**??**) in matrix form

$$T^a_{\ bk} = {e^a}_i {T^i}_{jk} {e^j}_b = \nabla_k {e^a}_j {e^j}_b, \tag{50}$$

$$T^a_{\ bk} = {e^a}_i {T^i}_{jk} {e^j}_b = -{e^a}_i \nabla_k {e^i}_b. \tag{51}$$

These relationships enable the dependence between the infinitesimal rotation $d\chi_{ab} = -d\chi_{ba}$ of the vector ${e^a}_i$ at infinitesimal translations ds_a to be established. In fact, by (50) and (51), we have

$$d{\chi^a}_b = T^a_{\ bk} dx^k = D{e^a}_j {e^j}_b, \tag{52}$$

$$d{\chi^a}_b = T^a_{\ bk} dx^k = -{e^a}_i D{e^i}_b. \tag{53}$$

where D is the absolute differential [29] with respect to the Christoffel symbols Γ^i_{jk}. Using (52), we can form the invariant quadratic form $d\tau^2 = d{\chi^a}_b d{\chi^b}_a$ to arrive at the Killing-Cartan metric

$$d\tau^2 = d{\chi^a}_b d{\chi^b}_a = T^a_{\ bk} {T^b}_{an} dx^k dx^n = -D{e^a}_i D{e^i}_a \tag{54}$$

with the metric tensor

$$H_{kn} = T^a_{\ bk} {T^b}_{an}. \tag{55}$$

Unlike metric (49), the metric (54) is specified on the rotations group $O(3.1)$ that acts on the manifold of the "rotational coordinates"${e^a}_i$.

Let us now introduce the covariant derivative

$$\overset{*}{\nabla}_m = \nabla_m + T_m, \tag{56}$$

where T_m is the matrix $T^a_{\ bm}$ with discarded matrix indices. We will regard the components of the derivative as generators of the rotations group $O(3.1)$. Applying this operator to the tetrad e^i that forms the manifold of "angular coordinates"of the A_4 geometry, we will arrive at

$$\overset{*}{\nabla}_m e^i = \nabla_m e^i + T_m e^i = 0, \tag{57}$$

hence

$$T_m = -e_i \nabla_m e^i. \tag{58}$$

It is interesting to note that, just as in (46) we have defined six "angular coordinates"${e^i}_a$ through the four translational coordinates x^i, so in (5.121) we can define 24 "supercoordinates"T^a_{bm} through the six coordinates ${e^i}_a$.

It follows from (57) that

$$\nabla_m e^i = -T_m e^i. \tag{59}$$

Recall that in the relationships (57)-(59) we have defined through ∇_m the covariant derivative with respect to Γ^i_{jk}. We will now take the covariant derivative ∇_k of the relationships (59)

$$\nabla_k \nabla_m e^i = -\nabla_k (T_m e^i) = -(\nabla_k T_m e^i + T_m \nabla_k e^i) =$$
$$= -(\nabla_k T_m e^i + T_m e^i e_i \nabla_k e^i).$$

Using (58), we will rewrite this expression as follows

$$\nabla_k \nabla_m e^i = -(\nabla_k T_m - T_m T_k) e^i.$$

Alternating this expression in the indices k and m gives

$$\nabla_{[k} \nabla_{m]} e^i = \frac{1}{2} R_{km} e^i, \tag{60}$$

where

$$R_{km} = 2\nabla_{[m} T_{k]} + [T_m, T_k]. \tag{61}$$

Introducing in equations (61) the matrix indices (the fibre indices), we will obtain the structural equation of the group $O(3.1)$

$$R^a{}_{bkm} = 2\nabla_{[m} T^a{}_{|b|k]} + 2T^a{}_{c[m} T^c{}_{|b|k]}. \tag{B}$$

It is easily seen that the structural equations of the rotations group (B) coincide with the second of Cartan's structural equations (61) of the geometry A_4.

In this case the quantities T^a_{bk} and $R^a{}_{bkm}$ vary in the rotations group $O(3.1)$ following the law

$$T^{a'}_{b'k} = \Lambda_a{}^{a'} T^a_{bk} \Lambda^b{}_{b'} + \Lambda_a{}^{a'} \Lambda^a{}_{b',k}, \tag{62}$$

and appear as the potentials of the gauge field $R^a{}_{bkm}$ of the rotations group $O(3.1)$. In the process, the gauge field of the group $O(3.1)$ obeys the formula

$$R^{a'}_{b'km} = \Lambda_a{}^{a'} R^a{}_{bkm} \Lambda^b{}_{b'}. \tag{63}$$

Note that the structural functions of the rotations group of A_4 geometry are the components of the curvature tensor $R^a{}_{bkm}$. It can be shown that the structural functions $R^a{}_{bkm}$ of the rotations group $O(3.1)$ satisfy the Jacobi identity

$$\nabla_{[n} R^a{}_{|b|km]} + R^c{}_{b[km} T^a{}_{|c|n]} - T^c{}_{b[n} R^a{}_{|c|km]} = 0, \tag{D}$$

which, at it was shown in the previous section, are at the same time the second Bianchi identities of the A_4 space.

Let us introduce the dual Riemann tensor

$$\overset{*}{R}_{ijkm}= \frac{1}{2}\varepsilon^{sp}{}_{km}R_{ijsp}, \tag{64}$$

where $\varepsilon^{sp}{}_{km}$ is the completely skew-symmetrical Levi-Chivita tensor. Then the equations (D) can be written as

$$\nabla_n \overset{*}{R}{}^a{}_b{}^{kn} + \overset{*}{R}{}^c{}_b{}^{kn} T^a_{cn} - T^c_{bn} \overset{*}{R}{}^a{}_c{}^{kn} = 0 \tag{65}$$

or, if we drop the matrix indices, as

$$\nabla_n \overset{*}{R}{}^{kn} + \overset{*}{R}{}^{kn} T_n - T_n \overset{*}{R}{}^{kn} = 0. \tag{66}$$

3 Structural equations of A_4 geometry in the form of expanded, completely geometrized Einstein-Yang-Mills set of equations

Einstein believed that one of the main problems of the unified field theory was the geometrization of the energy-momentum tensor of matter on the right-hand side of his equations. This problem can be solved if we use as the space of events the geometry of absolute parallelism and the structural Cartan equations for this geometry.

In fact, folding the equations (B), written as

$$R^i{}_{jkm} + 2\nabla_{[k}T^i_{|j|m]} + 2T^i_{s[k}T^s_{|j|m]} = 0 \tag{67}$$

in indices i and k, gives

$$R_{jm} = -2\nabla_{[i}T^i_{|j|m]} - 2T^i_{s[i}T^s_{|j|m]}. \tag{68}$$

If then we fold the equations (68) with the metric tensor g^{jm}, we have

$$R = -2g^{jm}(\nabla_{[i}T^i_{|j|m]} + 2T^i_{s[i}T^s_{|j|m]}). \tag{69}$$

Forming, using (68) and (69), the Einstein tensor

$$G_{jm} = R_{jm} - \frac{1}{2}g_{jm}R,$$

we obtain the equations

$$R_{jm} - \frac{1}{2}g_{jm}R = \nu T_{jm}, \tag{70}$$

which are similar to Einstein's equations, but with the geometrized right-hand side defined as

$$\begin{aligned} T_{jm} = -\frac{2}{\nu}\{(\nabla_{[i}T^i{}_{|j|m]} + T^i{}_{s[i}T^s{}_{|j|m]}) - \\ -\frac{1}{2}g_{jm}g^{pn}(\nabla_{[i}T^i{}_{|p|n]} + T^i{}_{s[i}T^s{}_{|p|n]})\} \end{aligned} \tag{71}$$

Using the notation

$$P_{jm} = (\nabla_{[i}T^i{}_{|j|m]} + T^i{}_{s[i}T^s{}_{|j|m]})$$

then, by (71), we have

$$T_{jm} = -\frac{2}{\nu}(P_{jm} - \frac{1}{2}g_{jm}g^{pn}P_{pn}). \tag{72}$$

Tensor (72) has parts that are both symmetrical and skew-symmetrical in indices j and m, i.e.,

$$T_{jm} = T_{(jm)} + T_{[jm]}. \tag{73}$$

The left-hand side of the equations (70) is always symmetrical in indices j and m, therefore these equations can be written as

$$R_{jm} - \frac{1}{2}g_{jm}R = \nu T_{(jm)}, \tag{74}$$

$$T_{[jm]} = \frac{1}{\nu}(-\nabla_i \Omega^{..i}_{jm} - \nabla_m A_j - A_s \Omega^{..s}_{jm}) = 0, \tag{75}$$

where

$$A_j = T^i_{ji}. \tag{76}$$

Relationship (75) can be taken to be the equations obeyed by the torsion fields $\Omega^{..i}_{jm}$, which form the energy-momentum tensor (72).

In the case where the field T^i_{jk} is skew-symmetrical in all the three indices, we get

$$T_{ijk} = -T_{jik} = T_{jki} = -\Omega_{ijk}. \tag{77}$$

For such fields the equations (75) become simple, namely

$$\nabla_i \Omega^{..i}_{jm} = 0. \tag{78}$$

The energy-momentum tensor (72) is symmetrical in indices j, m and appears to be given by

$$T_{jm} = \frac{1}{\nu}(\Omega^{..i}_{sm}\Omega^{..s}_{ji} - \frac{1}{2}g_{jm}\Omega^{.ji}_{s}\Omega^{..s}_{ji}). \tag{79}$$

By (74), we have

$$T_{jm} = \frac{1}{\nu}(R_{jm} - \frac{1}{2}g_{jm}R). \tag{80}$$

Using (68), (77) and (79) gives

$$R_{jm} = \Omega^{..i}_{sm}\Omega^{..s}_{ji}, \tag{81}$$

$$R = g^{jm}\Omega^{..i}_{sm}\Omega^{..s}_{ji} = \Omega^{.ji}_{s}\Omega^{..s}_{ji}. \tag{82}$$

Substituting (81) and (82) into (80), we arrive at the energy-momentum tensor (79).

Through the field (77) we can define the pseudo-vector h_m as follows

$$\Omega^{ijk} = \varepsilon^{ijkm}h_m, \quad \Omega_{ijk} = \varepsilon_{ijkm}h^m, \tag{83}$$

where ε_{ijkm} is the fully skew-symmetrical Levi-Chivita symbol.

In terms of the pseudo-vector h_m we can write the tensor (79) as follows

$$T_{jm} = \frac{1}{\nu}(h_j h_m - \frac{1}{2}g_{jm}h^i h_i). \tag{84}$$

Substituting the relationships (83) into (78), we get

$$h_{m,j} - h_{j,m} = 0. \tag{85}$$

These equations have two solutions: the trivial one, where $h_m = 0$, and

$$h_m = \psi_{,m}, \tag{86}$$

where Ψ is a pseudo-scalar.

Writing the energy-momentum tensor (85) in terms of this pseudo-scalar, we will have

$$T_{jm} = \frac{1}{\nu}(\psi_{,j}\psi_{,m} - \frac{1}{2}g_{jm}\psi^{,i}\psi_{,i}). \tag{87}$$

Tensor (87) is the energy-momentum tensor of a pseudo-scalar field.

Let us now decompose the Riemann tensor R_{ijkm} into irreducible parts

$$R_{ijkm} = C_{ijkm} + g_{i[k}R_{m]j} + g_{j[k}R_{m]i} + \frac{1}{3}Rg_{i[m}g_{k]j}, \tag{88}$$

where C_{ijkm} is the Weyl tensor; the second and third terms are the traceless part of the Ricci tensor R_{jm} and R is its trace.

Using the equations (70), written as

$$R_{jm} = \nu\left(T_{jm} - \frac{1}{2}g_{jm}T\right), \tag{89}$$

we will rewrite the relationship (88) as

$$R_{ijkm} = C_{ijkm} + 2\nu g_{[k(i}T_{j)m]} - \frac{1}{3}\nu T g_{i[m}g_{k]j}, \tag{90}$$

where T is the tensor trace (72).

Now we introduce the tensor current

$$J_{ijkm} = 2g_{[k(i}T_{j)m]} - \frac{1}{3}Tg_{i[m}g_{k]j} \tag{91}$$

and represent the tensor (90) as the sum

$$R_{ijkm} = C_{ijkm} + \nu J_{ijkm}. \tag{92}$$

Substituting this relationship into the equations (67), we will arrive at

$$C_{ijkm} + 2\nabla_{[k}T_{|ij|m]} + 2T_{is[k}T^{s}_{|j|m]} = -\nu J_{ijkm}. \tag{93}$$

Equations (93) are the Yang-Mills equations with a geometrized source, which is defined by the relationship (91). In equations (93) for the Yang-Mills field we have the Weyl tensor C_{ijkm}, and the potentials of the Yang-Mills field are the Ricci rotation coefficients T^{i}_{jk}.

We now substitute the relationship (92) into the second Bianchi identities (D)

$$\nabla_{[n}R_{|ij|km]} + R^{s}_{j[km}T_{|is|n]} - T^{s}_{j[n}R_{|is|km]} = 0. \tag{94}$$

We thus arrive at the equations of motion

$$\nabla_{[n}C_{|ij|km]} + C^{s}_{j[km}T_{|is|n]} - T^{s}_{j[n}C_{|is|km]} = -\nu J_{nijkm} \tag{95}$$

for the Yang-Mills field C_{ijkm}, such that the source J_{nijkm} in them is given in terms of the current (91) as follows:

$$J_{nijkm} = \nabla_{[n}J_{|ij|km]} + J^{s}_{j[km}T_{|is|n]} - T^{s}_{j[n}R_{|is|km]}. \tag{96}$$

Using the geometrized Einstein equations (70) and the Yang-Mills equations (93), we can represent the structural Cartan equations (A) and (B) as an extended set of Einstein-Yang-Mills equations

$$\boxed{\begin{array}{cr} \nabla_{[k}e^{a}_{j]} + T^{i}_{[kj]}e^{a}{}_{i} = 0, & (A) \\ R_{jm} - \frac{1}{2}g_{jm}R = \nu T_{jm}, & (B.1) \\ C^{i}_{jkm} + 2\nabla_{[k}T^{i}_{|j|m]} + 2T^{i}_{s[k}T^{s}_{|j|m]} = -\nu J^{i}{}_{jkm}, & (B.2) \end{array}} \tag{97}$$

in which the geometrized sources T_{jm} and J_{ijkm} are given by (72) and (91).

For the case of Einstein's vacuum the equations (97) are much simpler

$$\boxed{\begin{array}{cr} \nabla_{[k}e^{a}_{j]} + T^{i}_{[kj]}e^{a}{}_{i} = 0, & (i) \\ R_{jm} = 0, & (ii) \\ C^{i}{}_{jkm} + 2\nabla_{[k}T^{i}_{|j|m]} + 2T^{i}_{s[k}T^{s}_{|j|m]} = 0. & (iii) \end{array}} \tag{98}$$

The equations of motion (95) for the Yang-Mills field C_{ijkm} will then become

$$\nabla_{[n}C_{|ij|km]} + C^{s}_{j[km}T_{|is|n]} - T^{s}_{j[n}C_{|is|km]} = 0. \tag{99}$$

Equations (A) and $(B.2)$ can be written in matrix form

$$\nabla_{[k}e^{a}{}_{m]} - e^{b}{}_{[k}T^{a}{}_{|b|m]} = 0, \qquad (A)$$

$$C^{a}{}_{bkm} + 2\nabla_{[k}T^{a}{}_{|b|m]} + 2T^{a}{}_{f[k}T^{f}{}_{|b|m]} = -\nu J^{a}{}_{bkm}, \qquad (B.2)$$

where the current

$$J^{a}{}_{bkm} = 2g_{[k}{}^{(a}T_{b)m]} - \frac{1}{3}Tg^{a}{}_{[m}g_{k]b}, \tag{100}$$

is given by

$$T^{a}{}_{m} = \frac{1}{\nu}(R^{a}{}_{m} - \frac{1}{2}g^{a}{}_{m}R), \qquad (B.1)$$

$$m = 0,1,2,3, \quad a = 0,1,2,3.$$

By writing the equations (95) in matrix form, we have

$$\nabla_{[n}C^{a}{}_{|b|km]} + C^{c}{}_{b[km}T^{a}{}_{|c|n]} - T^{c}{}_{b[n}C^{a}{}_{|a|km]} = -\nu J^{a}{}_{nbkm}, \tag{101}$$

where

$$J^{a}{}_{nbkm} = \nabla_{[n}J^{a}{}_{|b|km]} + J^{c}{}_{b[km}T^{a}{}_{|c|n]} - T^{c}{}_{b[n}J^{a}{}_{|c|km]}. \tag{102}$$

Dropping the matrix indices in the matrix equations, we have

$$\nabla_{[k}e_{m]} - e_{[k}T_{m]} = 0, \tag{A}$$

$$C_{km} + 2\nabla_{[k}T_{m]} - [T_k, T_m] = -\nu J_{km}, \tag{B.2}$$

$$\nabla_n \overset{*}{C}{}^{kn} + [\overset{*}{C}{}^{kn}, T_n] = -\nu \overset{*}{J}{}^{k}, \tag{D}$$

where the dual matrices $\overset{*}{C}{}^{kn}$ and $\overset{*}{J}{}^{k}$ are given by

$$\overset{*}{C}{}^{kn} = \varepsilon^{knij} C_{ij},$$

$$\overset{*}{J}{}^{nk} = \varepsilon^{nkim} J_{im}, \tag{103}$$

$$\overset{*}{J}{}^{k} = \{\nabla_n \overset{*}{J}{}^{kn} + [\overset{*}{J}{}^{kn}, T_n]\}. \tag{104}$$

For the Einstein vacuum we have

$$R_{ijkm} = C_{ijkm} = \overset{*}{R}_{ijkm} = C_{ijkm}, \tag{105}$$

therefore the equations $(B.2)$ and (D) become simpler

$$C_{km} + 2\nabla_{[k}T_{m]} - [T_k, T_m] = 0, \tag{B.2}$$

$$\nabla_n C^{kn} + [C^{kn}, T_n] = 0. \tag{D}$$

Using the formalism of external differential forms, we can write the structural equations (A) and $(B.2)$ as follows:

$$de^a - e^b \wedge T^a_{\ b} = 0, \tag{A}$$

$$C^a_{\ b} + dT^a_{\ b} - T^a_{\ c} \wedge T^c_{\ b} = -\nu J^a_{\ b}, \tag{B.2}$$

and the equations (D) as

$$dC^a_{\ b} + C^a_{\ f} \wedge T^f_{\ b} - T^f_{\ b} \wedge C^a_{\ f} = -\nu N^a_{\ b}, \tag{D}$$

where

$$N^a_{\ b} = dJ^a_{\ b} + J^a_{\ f} \wedge T^f_{\ b} - T^f_{\ b} \wedge J^a_{\ f}. \tag{106}$$

Thus, the structural equations of A_4 geometry, written as (97), represent an extended set of Einstein-Yang-Mills equations with the gauge translations group T_4 defined on the base x^i with the structural equations (A), and with the gauge rotations group $O(3.1)$, defined in the fibre $e^i_{\ a}$ with the structural equations in the form of the geometrized equations $(B.1)$ and $(B.2)$.

4 Equations of geodesics of $\mathbf{A_4}$ spaces

The equations of geodesics for the geometry of absolute parallelism can be obtained from the conditions of parallel vector displacement

$$u^i = \frac{dx^i}{ds} \tag{107}$$

with respect to the connection of A_4 geometry

$$\Delta^i_{jk} = \Gamma^i_j + T^i_{jk} = e^i{}_a e^a{}_{j,k}. \tag{108}$$

In fact, we specialize the tetrad $e^i{}_a$ so that the vector $e^i{}_0$ would coincide with the tangent to the world line, i.e.,

$$e^i{}_0 = u^i = \frac{dx^i}{ds}. \tag{109}$$

From the relationships (5.27) for the vector (109) we have

$$\overset{*}{\nabla}_k u^i = u^i_{,k} + \Delta^i_{jk} u^j = 0 \tag{110}$$

or

$$\frac{\partial u^i}{\partial x^k} + \Gamma^i_{jk} u^j + T^i_{jk} u^j = 0. \tag{111}$$

Multiplying this by $u^k = dx^k/ds$ gives

$$\frac{du^i}{ds} + \Gamma^i_{jk} u^j u^k + T^i_{jk} u^j u^k = 0 \tag{112}$$

or, by (107),

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} + T^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{113}$$

These four equations ($i = 0, 1, 2, 3$) are the equations of geodesics of A_4 space. They are also the equations of motion for the origin O of tetrad $e^i{}_a$. Since in the equations (113) the Ricci rotation coefficients T^i_{jk} have both symmetrical and skew-symmetrical parts in indices j and k

$$T^i_{jk} = T^i_{(jk)} + T^i_{[jk]} =$$

$$= -\Omega^{\cdot\cdot i}_{jk} + g^{im}(g_{js}\Omega^{\cdot\cdot s}_{mk} + g_{ks}\Omega^{\cdot\cdot s}_{mj}), \tag{114}$$

$$T^i_{(jk)} = g^{im}(g_{js}\Omega^{\cdot\cdot s}_{mk} + g_{ks}\Omega^{\cdot\cdot s}_{mj}), \tag{115}$$

$$T^i_{[jk]} = -\Omega^{\cdot\cdot i}_{jk}, \tag{116}$$

we can write the equations (113) as

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} + T^i_{(jk)}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{117}$$

Considering the structure of the equality (115), we will write it in the form

$$T^i_{(jk)} = g^{im}(g_{js}\Omega^{\cdot\cdot s}_{mk} + g_{ks}\Omega^{\cdot\cdot s}_{mj}) = 2g^{im}\Omega_{m(jk)}, \tag{118}$$

hence the equations of geodesics for A_4 space can be represented as

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} + 2g^{im}\Omega_{m(jk)}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{119}$$

For the terns in (118) we can introduce the following notation:

$$\Omega^{\cdot i}_{k\cdot j} = g^{im}g_{ks}\Omega^{\cdot\cdot s}_{jm}, \quad \Omega^i_{\cdot jk} = g^{im}g_{ks}\Omega^{\cdot\cdot s}_{mj},$$

then the contorsion tensor T^i_{jk} for space A_4 will become

$$T^i_{jk} = -\Omega^{..i}_{jk} - \Omega^{.i}_{k.j} + \Omega^i_{.jk}, \tag{120}$$

where

$$-\Omega^{.i}_{k.j} = \Omega^i_{.jk},$$

whence

$$T^i_{jk} = -\Omega^{..i}_{jk} + 2\Omega^i_{.jk}. \tag{121}$$

The covariant differential of an arbitrary vector v^i with respect to the connection (108) for parallel displacement from point x^i to point $x^i + dx^i$ becomes

$$\delta v^i = dv^i + \Delta^i_{jk} dx^j = 0. \tag{122}$$

If at an arbitrary point x^i of A_4 space we have two linear elements δx^i and dx^i and make a parallel translation of δx^i along the element dx^i, then for the final point we will have [30]

$$x^i + dx^i + \delta x^i - \Delta^i_{jk}\delta x^k dx^j = x^i + dx^i + \delta x^i + d\delta x^i. \tag{123}$$

On the other hand, parallel translation of the vector dx^i along the vector δx^i gives

$$x^i + \delta x^i + dx^i - \Delta^i_{jk} dx^k \delta x^j = x^i + \delta x^i + dx^i + \delta dx^i. \tag{124}$$

Subtracting from the relationships (123) the equality (124), we get

$$\begin{aligned} d\delta x^i - \delta dx^i = -(\Delta^i_{jk}\delta x^k dx^j + \Delta^i_{jk} dx^k \delta x^j) = \\ -(\Delta^i_{jk} - \Delta^i_{kj})\delta x^k dx^j = -2\Delta^i_{[jk]}\delta x^k dx^j = \\ = 2\Omega^{..s}_{jk}\delta x^k dx^j = -2\Omega^{..s}_{jk}\delta x^j dx^k. \end{aligned} \tag{125}$$

Let us now consider the variation of the integral

$$\int_a^b L(x^i, u^i) ds, \tag{126}$$

where u^i is given by the relationship (107). We will write (125) as

$$\delta dx^i = d\delta x^i + 2\Omega^{..s}_{jk}\delta x^j dx^k. \tag{127}$$

Then at each point of the extremum we have

$$\delta u^i = \delta \frac{dx^i}{ds} = \frac{d}{ds}\delta x^i + 2\Omega^{..i}_{jk}\delta x^j \frac{dx^k}{ds}. \tag{128}$$

Applying a common variational procedure to the integral (126), we get

$$\begin{aligned} \int_a^b \delta L(x^i, u^i) ds = \\ \int_a^b \left(L(x^i + \delta x^i, u^i + \delta u^i) - L(x^i, u^i) \right) ds = \\ = \int_a^b \left(\frac{\partial L}{\partial x^i}\delta x^i + \frac{\partial L}{\partial u^i}\delta u^i \right) ds = 0. \end{aligned} \tag{129}$$

Substituting here the relationship (128) gives

$$\int_a^b \left(\frac{\partial L}{\partial x^i} \partial x^i + \frac{\partial L}{\partial u^i} \frac{d}{ds} \partial x^i + \frac{\partial L}{\partial u^i} 2\Omega_{jk}^{\cdot\cdot i} \partial x^j u^k \right) ds = 0.$$

We now integrate by parts the second term here to obtain

$$\int_a^b \left(\frac{\partial L}{\partial x^i} - \frac{d}{ds} \frac{\partial L}{\partial u^i} + 2\Omega_{ik}^{\cdot\cdot j} \frac{\partial L}{\partial u^j} u^k \right) \partial x^i = 0$$

or, since ∂x^i is arbitrary, we arrive at [30]

$$\frac{d}{ds} \frac{\partial L}{\partial u^i} - \frac{\partial L}{\partial x^i} + 2\Omega_{ki}^{\cdot\cdot j} \frac{\partial L}{\partial u^j} u^k = 0. \tag{130}$$

Let now

$$L = (g_{ik} u^i u^k)^{1/2}, \tag{131}$$

along the extremum $L = 1$ by the relationship

$$g_{ik} u^i u^k = u^i u_i = 1.$$

Substituting the Lagrangian (131) into equations (130) gives

$$g_{mi} \frac{du^i}{ds} + \Gamma_{mjk} u^j u^k + 2\Omega_{mj}^{\cdot\cdot s} g_{sk} u^k u^j = 0. \tag{132}$$

Multiplying this relationship by g^{im}, we get

$$\frac{du^i}{ds} + \Gamma_{kj}^i u^j u^k + 2g^{im} g_{ks} \Omega_{mj}^{\cdot\cdot s} u^j u^k = 0$$

or

$$\frac{du^i}{ds} + \Gamma_{kj}^i u^j u^k + 2g^{im} \Omega_{m(jk)} u^j u^k = 0. \tag{133}$$

We have thus obtained, using the variational principle, the equations of the geodesics in the form (119). Consider now the equations that describe the variation of the orientation of the tetrad $e^i{}_a$ as it moves according to the equations of the geodesics (133). We will rewrite the equations (**??**) as

$$\partial_k e^i{}_a + \Delta_{jk}^i e^j{}_a = 0$$

or

$$de^i{}_a + \Delta_{jk}^i e^j{}_a dx^k = 0. \tag{134}$$

Dividing these equations by ds yields

$$\frac{de^i{}_a}{ds} + \Delta_{jk}^i e^j{}_a \frac{dx^k}{ds} = 0. \tag{135}$$

Further, taking the second derivative $d^2 e^i{}_a / ds^2$, we will have

$$\frac{d}{ds} \left(\frac{de^i{}_a}{ds} \right) = \frac{d}{ds} \left(\frac{\partial e^i{}_a}{\partial x^k} \frac{dx^k}{ds} \right) = \frac{\partial^2 e^i{}_a}{\partial x^m \partial x^k} \frac{dx^k}{ds} \frac{dx^m}{ds} + \frac{\partial e^i{}_a}{\partial x^k} \frac{d^2 x^k}{ds^2}. \tag{136}$$

Since

$$\frac{\partial^2 e^i{}_a}{\partial x^m \partial x^k} = \frac{\partial}{\partial x^m}(-\Delta^i_{ak}) = \Delta^i_{jk,m} e^j{}_a -$$
$$-\Delta^i_{sk}(-\Delta^s_{jm} e^j{}_a) = (-\Delta^i_{jk,m} + \Delta^i_{sk}\Delta^s_{jm}) e^j{}_a$$

and

$$\frac{\partial e^i{}_a}{\partial x^k}\frac{d^2 x^k}{ds^2} = \Delta^i_{js}\Delta^s_{km}\frac{dx^k}{ds}\frac{dx^m}{ds} e^j{}_a,$$

we have

$$\frac{d^2 e^i{}_a}{ds^2} + (\Delta^i_{jk,m} - \Delta^i_{sk}\Delta^s_{jm} - \Delta^i_{js}\Delta^s_{km})\frac{dx^k}{ds}\frac{dx^m}{ds} e^j{}_a = 0. \tag{137}$$

Substituting here the sum (108), we have

$$\frac{d^2 e^i{}_a}{ds^2} + (\Gamma^i_{jk,m} + T^i_{jk,m} - \Gamma^i_{sk}\Gamma^s_{jm} - \Gamma^i_{sk}T^s_{jm} -$$
$$-T^i_{sk}\Gamma^s_{jm} - T^i_{sk}T^s_{jm} - \Gamma^i_{js}\Gamma^s_{km} - T^i_{js}\Gamma^s_{km} -$$
$$-\Gamma^i_{js}T^s_{km} - T^i_{js}T^s_{km})\frac{dx^k}{ds}\frac{dx^m}{ds} e^j{}_a = 0. \tag{138}$$

Since independent equations (138) (for three Euler's angles and three pseudo-Euclidean angles) describe the variation of the orientation of tetrad $e^i{}_a$ as it moves from the origin O according to the equations of geodesics (133).

In A_4 spaces, where the metric is flat

$$g_{ik} = \eta_{ik} = \mathrm{diag}(1 - 1 - 1 - 1), \tag{139}$$

the Christoffel symbols $\Gamma^i{}_{js}$ vanish and the equations (138) become

$$\frac{d^2 e^i{}_a}{ds^2} + (T^i_{jk,m} - T^i_{sk}T^s_{jm} - T^i_{js}T^s_{km})\frac{dx^k}{ds}\frac{dx^m}{ds} e^j{}_a = 0, \tag{140}$$

and the equations of geodesics (112) will become

$$\frac{d^2 x^i}{ds^2} + T^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{141}$$

We now introduce the tensor of the four-dimensional angular velocity of rotation tetrads $e^i{}_a$ [33]

$$\Omega_{ij} = T_{ijk}\frac{dx^k}{ds} = -\frac{de_{ia}}{ds} e^a{}_j = \frac{de_{ja}}{ds} e^a{}_i \tag{142}$$

with the symmetry properties

$$\Omega_{ij} = -\Omega_{ji}, \tag{143}$$

determined by the symmetry (**??**), for which the Ricci rotation coefficients hold.

Using (142), we will write the equations (141) and (140) as

$$\frac{d^2 x^i}{ds^2} + \Omega^i{}_j\frac{dx^j}{ds} = 0, \tag{144}$$

$$\frac{d\Omega^i{}_j}{ds} - T^i{}_{jk,m}\frac{dx^k}{ds}\frac{dx^m}{ds} + T^i{}_{js}\Omega^s{}_k\frac{dx_k}{ds} = 0. \qquad (145)$$

The skew-symmetric matrix (143) can be represented as

$$\Omega_{ij} = \begin{pmatrix} 0 & \Omega_{01} & \Omega_{02} & \Omega_{03} \\ \Omega_{10} & 0 & \Omega_{12} & \Omega_{13} \\ \Omega_{20} & \Omega_{21} & 0 & \Omega_{23} \\ \Omega_{30} & \Omega_{31} & \Omega_{32} & 0 \end{pmatrix} \qquad (146)$$

Let us now give a physical interpretation of the components of the matrix (146). We multiply the equations (144) by the mass m and rewrite them as

$$m\frac{d^2x_i}{ds^2} + m\Omega_{ij}\frac{dx^j}{ds} = 0. \qquad (147)$$

If the condition (139) holds, there equations can be represented as

$$m\frac{du_i}{ds_o} + m\Omega_{ij}\frac{dx^j}{ds_o} = 0, \qquad (148)$$

where

$$ds_o = (\eta_{ik}dx^i dx^k)^{1/2} \qquad (149)$$

is the pseudo-Euclidean metric and $u_i = dx_i/ds_o$.

We represent the equations (148) in the form

$$m\frac{du_i}{ds_o} = -mT_{i(jk)}\frac{dx^j}{ds_o}\frac{dx^k}{ds_o}, \qquad (150)$$

where the part of T symmetric in indices j and k is given by (115).

Assuming that motion governed by the equations (150) is nonrelativistic ($v/c \ll 1$), we will write the three-dimensional part of these equations as

$$m\frac{du_\alpha}{ds_o} = -mT_{\alpha(ok)}\frac{dx^o}{ds_o}\frac{dx^k}{ds_o} - 2mT_{\alpha(\beta k)}\frac{dx^\beta}{ds_o}\frac{dx^k}{ds_o} \qquad (151)$$

or, from the relationship (142), as

$$m\frac{du_\alpha}{ds_o} = -m\Omega_{\alpha o}\frac{dx^o}{ds_o} - 2m\Omega_{\alpha\beta}\frac{dx^\beta}{ds_o}. \qquad (152)$$

Since in the nonrelativistic approximation

$$ds_o = cdt, \quad u_a = \frac{v_\alpha}{c},$$

and $dx_o = cdt$, the equations (152) can be written as

$$m\frac{dv_\alpha}{dt} = -mc^2\Omega_{\alpha o} - 2mc^2\Omega_{\alpha\beta}\frac{1}{c}\frac{dx^\beta}{dt}. \qquad (153)$$

It is known from classical mechanics that the nonrelativistic equations of motion of the origin O of a three-dimensional accelerated reference frame under inertia forces alone have the form [34]

$$\frac{d}{dt}(m\mathbf{v}) = m\left(-\mathbf{W} + 2[\mathbf{v}\omega]\right), \tag{154}$$

where $\mathbf{W}$ is the vector of translational acceleration, and ω is the vector of the three-dimensional angular velocity of rotation of the accelerated reference frame.

We write these equations as

$$\frac{d}{dt}(mv_\alpha) = m\left(-W_{\alpha o} + 2\omega_{\alpha\beta}\frac{dx^\beta}{dt}\right), \tag{155}$$

where $\mathbf{W} = (W_{10}, W_{20}, W_{30})$,

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha} = -\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \tag{156}$$

$$\omega = (\omega_1, \omega_2, \omega_3),$$

and comparing these with (154), we obtain

$$\Omega_{10} = \frac{W_1}{c^2}, \quad \Omega_{20} = \frac{W_2}{c^2}, \quad \Omega_{30} = \frac{W_3}{c^2},$$

$$\Omega_{12} = -\frac{\omega_3}{c}, \quad \Omega_{13} = \frac{\omega_2}{c}, \quad \Omega_{23} = -\frac{\omega_1}{c}.$$

Therefore, the matrix (146) in this case has the form

$$\Omega_{ij} = \frac{1}{c^2}\begin{pmatrix} 0 & -W_1 & -W_2 & -W_3 \\ W_1 & 0 & -c\omega_3 & c\omega_2 \\ W_2 & c\omega_3 & 0 & -c\omega_1 \\ W_3 & -c\omega_2 & c\omega_1 & 0 \end{pmatrix} \tag{157}$$

It is seen from this matrix that the four-dimensional rotation of the tetrad $e^i{}_a$, caused by the torsion of the A_4 spaces, gives rise in physics to inertia fields associated with translational and rotational accelerations.